# ON STARLIKE FUNCTIONS

J. B. TWOMEY

**1. Introduction.** A function

$$f(z) = z + \sum_2^\infty a_n z^n \tag{1}$$

regular and univalent in $|z| < 1$ is said to be starlike there if it maps $|z| < 1$ onto a domain starshaped with respect to the origin. A regular function (1) is starlike in $|z| < 1$ if and only if $\operatorname{Re}[zf'(z)/f(z)] > 0$ in $|z| < 1$.

It was shown by the author in [2] that if $f$ is starlike then, for fixed arg $z$,

$$\left|\frac{zf'(z)}{f(z)}\right| \leqq \frac{(1 + o(1)) \log(4/r) \, |f(z)|}{(1 - r) \log(1/(1 - r))} \tag{2}$$

as $|z| = r \to 1$, and further that (2) is best possible in general as far as the radial rate of growth of $zf'(z)/f(z)$ is concerned. We remark in passing that the problem of finding the best possible lower bound for $|zf'(z)/f(z)|$ in terms of $|f(z)|$ seems to be open.

In this note, using a different method than that in [2], we obtain a sharpened form of (2). We prove in fact the following theorem.

THEOREM. *Let $f(z) = z + \sum_2^\infty a_n z^n$ be starlike in $|z| < 1$. Then*

$$\left|\frac{zf'(z)}{f(z)}\right| \leqq \frac{r \log \dfrac{(1 + r)^2}{r} \, |f(z)|}{(1 - r) \log \dfrac{1 + r}{1 - r}} + 1 \tag{3}$$

*for $|z| = r < 1$. Equality holds for the starlike function $f(z) = z(1 - z)^{-2}$ with $z = r$.*

For univalent functions of the form (1), it is well known (see, for example, [1, p. 5]) that

$$\frac{1 - r}{1 + r} \leqq \left|\frac{zf'(z)}{f(z)}\right| \leqq \frac{1 + z}{1 - r} \tag{4}$$

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for $|z| = r < 1$, and it may be of interest to note that the right-hand inequality of (4) is implied by (3) for starlike functions, since $|f(z)| \leqq r(1-r)^{-2}$ for normalized univalent $f$ [loc. cit.].

PROOF OF THEOREM.

2. We begin by proving that

$$\left|\frac{1+z}{1-z}\right| \leqq \frac{2r \log \dfrac{1+r}{|1-z|}}{(1-r)\log \dfrac{1+r}{1-r}} + 1 \tag{5}$$

for $|z| = r < 1$.

The function $(x-1)/\log x$ is an increasing function of $x$ in $x > 1$. Hence, for $u \geqq v > 1$

$$((u-1)/\log u)\log v + 1 \geqq ((v-1)/\log v)\log v + 1 = v.$$

This yields (5) for the case

$$\left|\frac{1+z}{1-z}\right| > 1$$

on putting

$$u = (1+r)/(1-r), \qquad v = \left|\frac{1+z}{1-z}\right| \quad (|z| = r < 1).$$

For

$$\left|\frac{1+z}{1-z}\right| \leqq 1$$

(5) is trivial, and we have thus established (5).

3. We now show that (3) follows easily from (5) and the representation theorem for starlike functions.

Since $\operatorname{Re}[zf'(z)/f(z)] > 0$ when $f$ is starlike, it is well known that we may write

$$\frac{zf'(z)}{f(z)} = \frac{1}{2\pi}\int_0^{2\pi} \frac{1+ze^{-it}}{1-ze^{-it}}\, d\mu(t) \tag{6}$$

where $\mu(t)$ increases and $\mu(2\pi) - \mu(0) = 2\pi$. From (6), by integration and the addition of $\log (1+r)^2$ to the resulting equation, we easily obtain

$$\log \frac{(1+r)^2}{r}\,|f(z)| = \frac{1}{\pi}\int_0^{2\pi} \log \frac{1+r}{|1-ze^{-it}|}\, d\mu(t) \tag{7}$$

on taking real parts.

From (6), (5) (with $z$ replaced by $ze^{-it}$) and (7) it now follows that

$$\left|\frac{zf'(z)}{f(z)}\right| \leqq \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{1+ze^{-it}}{1-ze^{-it}}\right| d\mu(t)$$

$$\leqq \frac{1}{2\pi}\int_0^{2\pi}\left\{\frac{2r\log\dfrac{1+r}{|1-ze^{-it}|}}{(1-r)\log\dfrac{1+r}{1-r}}+1\right\}d\mu(t)$$

$$= \frac{r\log\dfrac{(1+r)^2}{r}|f(z)|}{(1-r)\log\dfrac{1+r}{1-r}}+1.$$

This proves (3).

It is readily verified that we have equality in (3) for $f(z)=z(1-z)^{-2}$ with $z=r$ and we have thus completed the proof of the theorem.

I wish to thank the referee for suggesting a simplification of the proof of (5).

## References

1. W. K. Hayman, *Multivalent functions*, Cambridge Tracts in Math. and Math. Phys., no. 40, Cambridge Univ. Press, Cambridge, 1958. MR **21** #7302.

2. J. B. Twomey, *On the derivative of a starlike function*, J. London Math. Soc. (to appear).

University of South Florida