THE n-SEPARATED-ARC PROPERTY FOR HOMEOMORPHISMS

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ABSTRACT. Let f be a function defined in the open unit disk D whose range is in the Riemann sphere W, and let C denote the unit circle. We show that if f is a homeomorphism of D onto a Jordan domain, then the set of points $p \in C$ at which f has the n-separated-arc property ($n \geqq 2$) is a subset of the set of ambiguous points of f and is thus countable.

Let p be a point on C. If σ is an arc in D for which $\sigma \cup \{p\}$ is a Jordan arc, then σ is said to be an *arc at* p and the *cluster set* $C(f, p, \sigma)$ *of* f *at* p *along* σ is defined to be the set of all points $w \in W$ for which there exists a sequence $\{z_k\}$ of points on σ with $z_k \to p$ and $f(z_k) \to w$. We say that f possesses the *n-separated-arc property at p*, for some integer n ($n \geqq 2$), if there exist n mutually disjoint arcs $\sigma_1, \cdots, \sigma_n$ at p for which the intersection of all n of the sets $C(f, p, \sigma_j)$ $(j=1, \cdots, n)$ is empty while the intersection of any $n-1$ of them is nonempty. A point $p \in C$ at which f has the 2-separated-arc property is called an *ambiguous point of* f.

By a *Jordan domain* we mean an open connected subset of W whose boundary is a Jordan curve. Since each analytic homeomorphism of D onto a Jordan domain is necessarily continuous on $D \cup C$, the set of points at which such a function has the n-separated-arc property ($n \geqq 2$) is empty. The purpose of this note is to determine the nature of this set for nonanalytic homeomorphisms.

THEOREM. *Let f be a homeomorphism of the open unit disk D onto a Jordan domain U. If f has the n-separated-arc property ($n > 2$) at the point $p \in C$, then p is an ambiguous point of f.*

PROOF. Let $\sigma_1, \cdots, \sigma_n$ be n mutually disjoint arcs at p for which the intersection of all n of the sets $C(f, p, \sigma_j)$ $(j=1, \cdots, n)$ is empty while the intersection of any $n-1$ of them is nonempty. Then

$$C(f, p, \sigma_1) \cap C(f, p, \sigma_n) = \alpha_1 \cup \alpha_2,$$

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where α_1 and α_2 are closed connected subsets of the boundary ∂U of U with $\alpha_1 \cap \alpha_2 = \emptyset$ [note that one of the α_t $(t=1, 2)$ might be empty and one or both of them might be a singleton].

Without loss of generality, we assume that there exists a Jordan arc γ lying in D such that $\sigma_1 \cup \sigma_n \cup \gamma \cup \{p\}$ is a Jordan curve whose interior domain Δ contains the arcs $\sigma_2, \cdots, \sigma_{n-1}$. If no point $w \in \partial U$ is accessible (see e.g. [2, p. 168]) from $f(\Delta)$, then for at least one of the sets α_t $(t=1, 2)$, say α_1, we have the relations $\emptyset \neq \alpha_1 \subset C(f, p, \sigma)$ for each arc σ at p satisfying $\sigma \subset \Delta$. In particular, $\alpha_1 \subset C(f, p, \sigma_j)$ for each $j=1, \cdots, n$ in violation of the fact that the intersection of all n of these sets is empty. Thus, there exists a point $w^* \in \partial U$ which is accessible from $f(\Delta)$. This implies the existence of an arc σ^* at p with $\sigma^* \subset \Delta$ for which the cluster set $C(f, p, \sigma^*) = \{w^*\}$. Consequently,

$$C(f, p, \sigma^*) \cap C(f, p, \sigma_j) = \emptyset$$

for some j $(j=1, \cdots, n)$ and p is an ambiguous point of f as asserted in the theorem.

REMARK. According to Mathews [3, Example 1, p. 169], the converse of this theorem is not true, even for a differentiable homeomorphism.

In view of Bagemihl's ambiguous point theorem [1, Theorem 2, p. 380], we have the following result.

COROLLARY. *If f is a homeomorphism of the open unit disk D onto a Jordan domain U, then the set of points at which f has the n-separated-arc property* $(n \geqq 2)$ *is countable.*

We note that this result is sharp in that one can readily construct a nonanalytic homeomorphism of D onto D having infinitely many ambiguous points.

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