ON ABSOLUTE BOREL-TYPE METHODS OF SUMMABILITY

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1. Introduction. Suppose throughout that \( l, a_n \) \((n = 0, 1, \ldots)\) are arbitrary complex numbers, that \( \lambda > 0 \) and \( \mu \) is real, and that \( N \) is a nonnegative integer such that \( \lambda N + \mu \geq 1 \). Let \( s_{-1} = 0, s_n = \sum_{r=0}^{n} a_r; \)

\[
a_{\lambda, \mu}(x) = \sum_{n=N}^{\infty} \frac{a_n x^{\lambda n + \mu - 1}}{\Gamma(\lambda n + \mu)}, \quad s_{\lambda, \mu}(x) = \sum_{n=N}^{\infty} \frac{s_n x^{\lambda n + \mu - 1}}{\Gamma(\lambda n + \mu)}. \]

Borel-type methods of summability are defined as follows: The series \( \sum_{n=0}^{\infty} a_n \) is said to be

(i) summable \((B, \lambda, \mu)\) to \( l \), if \( s_{\lambda, \mu}(x) \) is finite for all \( x \geq 0 \) and \( \lambda e^{-x} s_{\lambda, \mu}(x) \rightarrow l \) as \( x \rightarrow \infty \);

(i)' summable \((B', \lambda, \mu)\) to \( l \), if \( a_{\lambda, \mu}(x) \) is finite for all \( x \geq 0 \) and \( \int_{0}^{y} e^{-x} a_{\lambda, \mu}(x) dx + s_{N-1} \rightarrow l \) as \( y \rightarrow \infty \);

(ii) absolutely summable \((B, \lambda, \mu)\), or summable \(|B, \lambda, \mu|\), to \( l \), if the series is summable \((B, \lambda, \mu)\) to \( l \) and \( e^{-x} s_{\lambda, \mu}(x) \) is of bounded variation on \([0, \infty)\);

(ii)' absolutely summable \((B', \lambda, \mu)\), or summable \(|B', \lambda, \mu|\), to \( l \), if the series is summable \((B, \lambda, \mu)\) to \( l \) and \( \int_{0}^{y} e^{-x} a_{\lambda, \mu}(x) dx \) is of bounded variation on \([0, \infty)\).

Note that the methods \((B, 1, 1)\) and \((B', 1, 1)\) are respectively equivalent to the standard Borel exponential and integral methods \( B \) and \( B' \).

The object of this paper is to establish the following absolute summability analogue of a known inclusion theorem for ordinary Borel-type summability ([2, Result 1] and [1, Theorem 2]; see also [4]):

**Theorem.** If \( \alpha > \lambda \), the series \( \sum_{n=0}^{\infty} a_n \) is summable \(|B', \alpha, \beta|\) to \( l \), and \( a_{\lambda, \mu}(x) \) is finite for all \( x \geq 0 \), then the series is summable \(|B', \lambda, \mu|\) to \( l \).

It is known that [1, Lemma 4] \( a_{\lambda, \mu}(x) \) is finite for all \( x \geq 0 \) if and only if \( s_{\lambda, \mu}(x) \) is finite for all \( x \geq 0 \); and that [3, Theorem 17] a series is summable \(|B', \lambda, \mu|\) to \( l \) if and only if it is summable \(|B, \lambda, \mu + 1|\) to \( l \). Hence \( "B'" \) may be replaced by \( "B" \) in the theorem.

2. Preliminary results.

**Lemma 1.** If \( \delta > 0 \) and a series is summable \(|B', \alpha, \beta|\) to \( l \) then it is

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summable \( |B', \alpha, \beta \pm \delta| \) to 1.

This lemma is known \([5]\).

**Lemma 2.** If \( \alpha > \lambda \) and \( \beta / \alpha \geq \mu / \lambda \), then there is a function \( \psi \), continuous on \((0, \infty)\), such that

\[
\frac{\Gamma(\alpha n + \beta)}{\Gamma(\lambda n + \mu)} = \int_0^\infty t^n \psi(t) \, dt \quad (n \geq N),
\]

\[
\int_0^\infty t^n |\psi(t)| \, dt = O\left(\frac{\Gamma(\alpha n + \beta)}{\Gamma(\lambda n + \mu)}\right) \quad (n \geq N),
\]

and, for any \( \delta > 0 \),

\[
u^{(\alpha - \lambda)} \psi(u^{\alpha - \lambda}) = O(e^{-\rho u}(u^{\lambda/2} + u^{-\sigma - \delta})) \quad (0 < u < \infty)
\]

where \( \rho = 1 - (\beta - \mu)/(\alpha - \lambda) \), \( \sigma = \beta - \alpha \mu / \lambda \), \( k = ((\alpha - \lambda)/\lambda)(\lambda/\alpha)^{(a-\lambda)} \).

**Proof.** Let \( h(s) = \Gamma(\alpha s + \beta)/\Gamma(\lambda s + \mu) \). Then by Stirling's theorem (see \([2, p. 129]\)), there is a positive constant \( C \) such that

\[
h(s) = e^{(\alpha \log \alpha - \log \alpha + \lambda) s + \beta - \mu}(C + O(1/|s|))
\]

when \( |s| \) is large and \( \Re s > -\mu / \lambda \). Since \( N > -\mu / \lambda \), it follows from the proof of Lemma 4 in \([2]\), with \( \sigma_0 = -\mu / \lambda \), \( \nu = N \), that there is a function \( \phi \), continuous on \((0, \infty)\), such that

\[
h(n) = \int_0^\infty t^{n-N} \phi(t) \, dt \quad (n \geq N);
\]

\[
\int_0^\infty t^{n-N} |\phi(t)| \, dt = O(h(n)) \quad (n \geq N);
\]

\[
t^{-N} \phi(t) = O(t^{\mu/\lambda - \delta/(\alpha - \lambda)}) = O(t^{-(\sigma + \delta)/(\alpha - \lambda)}) \quad \text{as} \quad t \to 0+;
\]

and

\[
t^{-N} \phi(t) \sim Ke^{-k t^{1/(\alpha - \lambda)} t^{-\rho + 1/2(\alpha - \lambda)}} \quad \text{as} \quad t \to \infty,
\]

where \( K \) is a positive constant.

Putting \( \psi(t) = t^{-N} \phi(t) \), we obtain the conclusions of Lemma 2.

3. **Proof of the theorem.** Let

\[
\gamma = \alpha / \lambda, \quad \rho = 1 - (\beta - \mu)/(\alpha - \lambda), \quad \sigma = \beta - \gamma \mu,
\]

\[
k = (\gamma - 1) \gamma^{\gamma/(1-\gamma)}, \quad \delta = (\gamma - 1)^2 / \gamma.
\]

By Lemma 1, we may suppose, without loss in generality that \( \beta \geq \gamma \mu \), i.e. that \( \sigma \geq 0 \).
The main hypotheses of the theorem are that

\[ (4) \quad \int_0^\infty e^{-y} | a_{\alpha,\beta}(y) | \, dy < \infty, \]

and that

\[ (5) \quad a_{\lambda,\mu}(x) \text{ is finite for all } x \geq 0. \]

Let \( \psi \) be the function specified in Lemma 2. Then, for \( 0 < x < \infty \),

\[ a_{\lambda,\mu}(x) = \sum_{n=N}^\infty \frac{a_n x^{\lambda n + \mu - 1}}{\Gamma(\alpha n + \beta)} \frac{\Gamma(\alpha n + \beta)}{\Gamma(\lambda n + \mu)} \int_0^\infty t^n \psi(t) \, dt \]

\[ = x^{\mu - 1 + (1 - \beta)/\gamma} \int_0^\infty t^{(1 - \beta)/\alpha} \psi(t) \, dt \sum_{n=N}^\infty \frac{a_n (x^{1/\gamma} t^{1/\alpha})^{\alpha n + \beta - 1}}{\Gamma(\alpha n + \beta)} \]

\[ = x^{\mu - 1 + (1 - \beta)/\gamma} \int_0^\infty t^{(1 - \beta)/\alpha} \psi(t) a_{\alpha,\beta}(x^{1/\gamma} t^{1/\alpha}) \, dt, \]

the inversion of sum and integral being legitimate since, by (2), there is a constant \( M \) such that

\[ \sum_{n=N}^\infty \left| \frac{a_n}{\Gamma(\alpha n + \beta)} \right| \int_0^\infty t^n \left| \psi(t) \right| \, dt < M \sum_{n=N}^\infty \left| \frac{a_n}{\Gamma(\lambda n + \mu)} \right| x^{\lambda n + \mu - 1}, \]

which is finite by (5).

Substitute \( t = x^{-\lambda} y^{\alpha}, \, dt = \alpha x^{-\lambda} y^{\alpha - 1} \, dy \) in the final integral in (6) to get

\[ a_{\lambda,\mu}(x) = \alpha x^{\mu - 1 - \lambda} \int_0^\infty y^{\alpha - \beta} a_{\alpha,\beta}(y) \psi(x^{-\lambda} y^{\alpha}) \, dy \quad (0 < x < \infty), \]

and hence

\[ \int_0^\infty e^{-x} \left| a_{\lambda,\mu}(x) \right| \, dx \]

\[ \leq \alpha \int_0^\infty \left| a_{\alpha,\beta}(y) \right| y^{\alpha - \beta} \, dy \int_0^\infty e^{-x y^{\alpha - \lambda}} \left| \psi(x^{-\lambda} y^{\alpha}) \right| \, dx. \]

Now substitute \( x = y^{1/\gamma - 1}, \, dx = (\gamma - 1) y^{1/\gamma - 2} \, dy \) in the inner integral on the right-hand side of (7) to get

\[ \int_0^\infty e^{-x} \left| a_{\lambda,\mu}(x) \right| \, dx \]

\[ \leq \alpha (\gamma - 1) \int_0^\infty \left| a_{\alpha,\beta}(y) \right| \, dy \int_0^\infty e^{-y^{1/\gamma - 1 - \sigma - 1} (y^{1/\gamma - \lambda} \sigma^{\lambda})} \left| \psi((y^{1/\gamma - \lambda} \sigma^{\lambda}) \right| \, dv. \]
Consequently, by (3), there is a constant $M_1$ such that
\begin{equation}
\int_0^\infty e^{-x} | a_{\lambda, \mu}(x) | \, dx \leq M_1 \int_0^\infty e^{-\nu} | a_{\alpha, \beta}(\nu) | \, I(\nu) \, d\nu
\end{equation}
where
\[ I(\nu) = \int_0^\infty e^{-(\nu^\gamma - \nu^k)\nu^q} \left\{ \left( \frac{\nu}{\nu^\gamma} \right)^{1/2} + \left( \frac{\nu}{\nu^\gamma} \right)^{-\delta} \right\} \nu^{\nu^\gamma - 1} \, d\nu. \]

Let $f(\nu) = \nu^\gamma - \nu^k, \ c = \gamma^{1/(1-\gamma)}$. Then $f(c) = f'(c) = 0, f(\nu) > 0$ when $\nu > 0, \ \nu \neq c$, and
\[ f(\nu)/(\nu - c)^2 \to f''(c)/2 = \gamma(\gamma - 1)\nu^{\gamma - 2}/2 \quad \text{as} \quad \nu \to c. \]

Hence there are positive constants $\rho, \ q, \ r$ such that
\[ f(\nu) \geq \rho, \ \nu^{-\gamma}f(\nu) \geq q \quad \text{when} \ 0 < \nu < c/2 \quad \text{or} \quad \nu > 3c/2; \]
and $f(\nu) \geq rv(\nu - c)^2$ when $c/2 < \nu < 3c/2$.

It follows that, for $y > 0$,
\begin{align*}
I(y) & \leq y^{1/2} \int_{c/2}^{3c/2} e^{-r(\nu^c - \nu^k)^2/\nu} \nu^{\nu^c - 3/2} \, d\nu + y^{1/2} \int_0^\infty e^{-\rho\nu^{1/2}} \nu^{\nu^c - 3/2} \, d\nu \\
& \quad + y^{-\delta} \int_{c/2}^{3c/2} \nu^{\theta - 1}\nu^{\nu^c} \nu^{\nu^c - 3/2} \, d\nu + y^{-\delta} \int_0^\infty e^{-\rho\nu^{1/2}} \nu^{\nu^c - 3/2} \, d\nu \\
& \leq 2 \left( \frac{c}{2} \right)^{-\delta - 3/2} y^{1/2} \int_{c/2}^{3c/2} \nu^{\nu^c} \, d\nu + y^{-\delta} \int_0^\infty e^{-\rho\nu^{1/2}} \nu^{\nu^c - 3/2} \, d\nu \\
& \quad + y^{-\delta} \left( \frac{3c}{2} \right)^{\delta - 1} + y^{-\delta} \delta/(\gamma - 1) \int_0^\infty e^{-\rho\nu^{1/2}} \nu^{\nu^c - 3/2} \, d\nu \\
& \leq M_2(1 + y^{-\delta} + y^{-\delta - \gamma + 1} + y^{-\delta - \gamma + 1})
\end{align*}
where $M_2$ is a constant; i.e.
\begin{equation}
I(y) = O(1) \quad (1 \leq y < \infty),
\end{equation}
and, since $\delta = (\gamma - 1)^2/\gamma < \gamma - 1$,
\begin{equation}
I(y) = O(y^{-\delta - \gamma + 1}) \quad (0 < y < 1).
\end{equation}

In virtue of (10), we have
\begin{align*}
I(y)a_{\alpha, \beta}(y) & = I(y) \sum_{n=N}^\infty \frac{a_n y^{\alpha n + \beta - 1}}{\Gamma(\alpha n + \beta)} \\
& = O(y^{-\delta - \gamma + 1 + \alpha N + \beta - 1}) = O(y^{\gamma(\alpha N + \beta - 1)}), \\
& = O(1) \quad (0 < y < 1).
\end{align*}
It follows from (4), (9) and (11) that
\[
\int_0^\infty e^{-y} |a_{\alpha,\beta}(y)| I(y) dy < \infty.
\]
Consequently, by (8),
\[
\int_0^\infty e^{-x} |a_{\lambda,\mu}(x)| dx < \infty,
\]
i.e. \(\sum_0^\infty a_n\) is summable \(|B, \lambda, \mu|\).

Further, by the inclusion theorem for ordinary Borel-type summability referred to in §1, the \(|B, \lambda, \mu|\) sum of the series \(\sum_0^\infty a_n\) is the same as its \(|B, \alpha, \beta|\) sum. This completes the proof.

References


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