NOTE ON A THEOREM OF NEHARI ON HANKEL FORMS

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Let Γ be a discrete ordered abelian group and let G be its Pontrjagin dual, both written in additive notation. It is known [**3**, Chapter 8] that G is compact and connected. We denote the set of nonnegative elements of Γ by Γ^+. Then a Hankel form on Γ is defined to be a complex bilinear form

$$A(a, b) = \sum_{\mu,\nu\in\Gamma^+} \phi(\mu + \nu)a(\mu)b(\nu), \tag{1}$$

where a, b and ϕ are functions on Γ^+. If

$$|A(a, b)| \leqq M < \infty$$

for all functions a and b with

$$\sum_{\mu\in\Gamma^+} |a(\mu)|^2 = \sum_{\nu\in\Gamma^+} |b(\nu)|^2 = 1, \tag{2}$$

then the Hankel form A is called bounded and M is called a bound of A. The purpose of this note is to prove the following criterion of boundedness of Hankel forms:

THEOREM. *The Hankel form* (1) *on* Γ *is bounded with a bound* M *if and only if there is a function* $f\in L^\infty(G)$ *with respect to the normalized Haar measure* dx *on* G *such that*

$$\|f\|_\infty \leqq M \tag{3}$$

and

$$\phi(\mu) = \int f(x)(x, \mu)dx, \qquad \mu \in \Gamma^+. \tag{4}$$

In particular this theorem implies that the least bound of A may be realized by the L^∞-norm of some f satisfying these conditions. This is exactly where the crux of the proof lies.

This theorem is due to Nehari [**2**] in case Γ is the additive group of integers and G is the circle group. His proof relies heavily on complex analysis and does not generalize to our case. Our technique is drawn from the theory of Hardy classes [**1**], [**3**].

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The sufficiency of the conditions (3) and (4) is immediate. In fact let a, b be given satisfying (2). Then by Plancherel's theorem, the functions α, β in the Hardy class H^2 (G) defined by the mean-convergent series

$$\alpha(x) = \sum_{\mu\in\Gamma^+} a(\mu)(x, \mu), \qquad \beta(x) = \sum_{\nu\in\Gamma^+} b(\nu)(x, \nu)$$

have L^2-norms equal to 1. Further

$$\begin{aligned} | A(a, b) | &= \left| \sum_{\mu,\nu\in\Gamma^+} \int f(x)(x, \mu + \nu)a(\mu)b(\nu)dx \right| \\ &= \left| \int f(x)\alpha(x)\beta(x)dx \right| \leqq \|f\|_\infty \|\alpha\|_2 \|\beta\|_2 \leqq M. \end{aligned}$$

Next we like to prove the necessity part of the theorem. Thus let A be a Hankel form on Γ bounded by M. For a function $g\in H^1(G)$, let the Fourier coefficients of g be

$$c(\mu) = \int g(x)\overline{(x, \mu)}dx, \qquad \mu \in \Gamma^+.$$

Then we claim that the series

$$\sum_{\mu\in\Gamma^+} \phi(\mu)c(\mu) \tag{5}$$

converges. In fact, let h and k be the inner and outer functions such that $g=hk$. Put $\alpha=hk^{1/2}$, $\beta=k^{1/2}$, the square root being taken in a continuous fashion. Then from the theory of Dirichlet algebras [1], α, $\beta\in H^2$, and $\|\alpha\|_2=\|\beta\|_2=\|g\|_1^{1/2}$. Denote the Fourier coefficients of α and β by $a(\mu)$ and $b(\nu)$ respectively. Then

$$\sum | a(\mu) |^2 = \sum | b(\nu) |^2 = \|g\|_1;$$

and (5) is reduced to $A(a, b)$, and is therefore convergent. Define the sum of (5) as Tg. T is clearly a linear functional on H^1. Since

$$| Tg | = | A(a, b) | \leqq M\|a\|_2\|b\|_2 = M\|g\|_1,$$

T is a bounded linear functional on H^1 with norm not exceeding M. By the Hahn-Banach theorem there is a norm-preserving extension of T to $L^1(G)$ which is given by some L^∞ function f:

$$Tg = \int f(x)g(x)dx, \qquad g \in L^1, \tag{6}$$

satisfying (3). The condition (4) is then a consequence of (6) by taking $g(x) = (x, \mu)$. The theorem is therewith proved.

References

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3. W. Rudin, *Fourier analysis on groups*, Interscience Tracts in Pure and Appl. Math., no. 12, Interscience, New York, 1962. MR **27** #2808.

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