NOTE ON A THEOREM OF NEHARI ON HANKEL FORMS

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Let $\Gamma$ be a discrete ordered abelian group and let $G$ be its Pontrjagin dual, both written in additive notation. It is known [3, Chapter 8] that $G$ is compact and connected. We denote the set of nonnegative elements of $\Gamma$ by $\Gamma^+$. Then a Hankel form on $\Gamma$ is defined to be a complex bilinear form

$$A(a, b) = \sum_{\mu, \nu \in \Gamma^+} \phi(\mu + \nu) a(\mu) b(\nu),$$

where $a, b$ and $\phi$ are functions on $\Gamma^+$. If

$$|A(a, b)| \leq M < \infty$$

for all functions $a$ and $b$ with

$$\sum_{\mu \in \Gamma^+} |a(\mu)|^2 = \sum_{\nu \in \Gamma^+} |b(\nu)|^2 = 1,$$

then the Hankel form $A$ is called bounded and $M$ is called a bound of $A$. The purpose of this note is to prove the following criterion of boundedness of Hankel forms:

**Theorem.** The Hankel form (1) on $\Gamma$ is bounded with a bound $M$ if and only if there is a function $f \in L^\infty(G)$ with respect to the normalized Haar measure $dx$ on $G$ such that

$$||f||_\infty \leq M$$

and

$$\phi(\mu) = \int f(x)(x, \mu) dx, \quad \mu \in \Gamma^+.$$

In particular this theorem implies that the least bound of $A$ may be realized by the $L^\infty$-norm of some $f$ satisfying these conditions. This is exactly where the crux of the proof lies.

This theorem is due to Nehari [2] in case $\Gamma$ is the additive group of integers and $G$ is the circle group. His proof relies heavily on complex analysis and does not generalize to our case. Our technique is drawn from the theory of Hardy classes [1], [3].

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103
The sufficiency of the conditions (3) and (4) is immediate. In fact let \( a, b \) be given satisfying (2). Then by Plancherel's theorem, the functions \( \alpha, \beta \) in the Hardy class \( H^2 (G) \) defined by the mean-convergent series

\[
\alpha(x) = \sum_{\mu \in \Gamma^+} a(\mu)(x, \mu), \quad \beta(x) = \sum_{\nu \in \Gamma^+} b(\nu)(x, \nu)
\]

have \( L^2 \)-norms equal to 1. Further

\[
| A(a, b) | = \left| \sum_{\mu, \nu \in \Gamma^+} \int f(x)(x, \mu + \nu)a(\mu)b(\nu)dx \right|
\]

\[
\leq \| f \|_\infty \| \alpha \|_2 \| \beta \|_2 \leq M.
\]

Next we like to prove the necessity part of the theorem. Thus let \( A \) be a Hankel form on \( \Gamma \) bounded by \( M \). For a function \( g \in H^1(G) \), let the Fourier coefficients of \( g \) be

\[
c(\mu) = \int g(x)(x, \mu)dx, \quad \mu \in \Gamma^+.
\]

Then we claim that the series

\[
(5) \sum_{\mu \in \Gamma^+} \phi(\mu)c(\mu)
\]

converges. In fact, let \( h \) and \( k \) be the inner and outer functions such that \( g = hk \). Put \( \alpha = h\sqrt{k} \), \( \beta = k\sqrt{\nu} \), the square root being taken in a continuous fashion. Then from the theory of Dirichlet algebras \([1]\), \( \alpha, \beta \in H^2 \), and \( \| \alpha \|_2 = \| \beta \|_2 = \| g \|_1^2 \). Denote the Fourier coefficients of \( \alpha \) and \( \beta \) by \( a(\mu) \) and \( b(\nu) \) respectively. Then

\[
\sum |a(\mu)|^2 = \sum |b(\nu)|^2 = \| g \|_1^2;
\]

and (5) is reduced to \( A(a, b) \), and is therefore convergent. Define the sum of (5) as \( Tg \). \( T \) is clearly a linear functional on \( H^1 \). Since

\[
| Tg | = | A(a, b) | \leq M\| a \|_2\| b \|_2 = M\| g \|_1,
\]

\( T \) is a bounded linear functional on \( H^1 \) with norm not exceeding \( M \). By the Hahn-Banach theorem there is a norm-preserving extension of \( T \) to \( L^1(G) \) which is given by some \( L^\infty \) function \( f \):

\[
(6) \quad Tg = \int f(x)g(x)dx, \quad g \in L^1,
\]
satisfying (3). The condition (4) is then a consequence of (6) by taking $g(x) = (x, \mu)$. The theorem is therewith proved.

**References**

