JORDAN HOMOMORPHISMS AND DERIVATIONS ON SEMISIMPLE BANACH ALGEBRAS

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1. **Introduction.** One may construct a Jordan homomorphism from one (associative) ring into another ring by taking the sum of a homomorphism and an antihomomorphism of the first ring into two ideals in the second ring with null intersection [6]. A number of authors have considered conditions on the rings that imply that every Jordan homomorphism, or isomorphism, is of this form [6], [3], [7], [10], [11]. We use a theorem of I. N. Herstein [3] (and without restrictions on the characteristic of the ring [10]) to show that a Jordan homomorphism from a ring onto a semisimple ring with identity and no one-dimensional irreducible (left) modules is the sum of a homomorphism and an antihomomorphism (Theorem 2.1). The ideals with null intersection are obtained from a disconnection of the structure space in a way similar to that used in the proof of Corollary 3.8 of [11, p. 446]. I am indebted to B. E. Johnson for drawing my attention to [11]. We give an example to show that the conclusion is false if the hypothesis that there be no one-dimensional irreducible modules is omitted. In Theorem 2.2 we show that for C^*-algebras the decomposition of Theorem 2.1 holds if the assumption that there is an identity is dropped.

I. N. Herstein has shown that a Jordan derivation on a prime ring not of characteristic 2 is a derivation [4]. We use this result to show that a continuous Jordan derivation on a semisimple Banach algebra is a derivation (Theorem 3.3). As an application of Theorem 3.3 we show that a hermitian operator *on* a C^*-algebra with identity is the sum of a left multiplication by a selfadjoint element in the algebra and a *-derivation.

2. **Jordan epimorphisms.** An additive map θ from a ring A to a ring B is called a Jordan homomorphism if $\theta(a \circ b) = \theta(a) \circ \theta(b)$ for all a and b in A where $x \circ y = xy + yx$ (we shall write all definitions and proofs for a ring of characteristic not equal to 2; the necessary changes for characteristic 2 are routine [10]). We take the dimension of an irreducible module over a ring to be the dimension of the module as a module over the centralizer [5, p. 24].

Let A be a semisimple ring with identity, and let Π be the set of all (left) primitive ideals in A with the Jacobson topology [5]. If Ψ

Received by the editors March 11, 1969.

and Φ are closed disjoint nonempty subsets of Π whose union is Π, and if $J=\bigcap\{P\in\Psi\}$ and $I=\bigcap\{P\in\Phi\}$, then $A=I\oplus J$. This result is known. We obtain $I+J=A$, because a maximal ideal in A is primitive, Ψ and Φ are closed, and $\Psi\cap\Phi=\varnothing$. That $I\cap J=\{0\}$ follows from the semisimplicity of A and $\Psi\cup\Phi=\Pi$.

2.1. THEOREM. *Let A and B be rings, and let A be semisimple with identity and no irreducible modules of dimension one. Let θ be a Jordan epimorphism from B to A. Then there are unique ideals I and J in A, a unique epimorphism ϕ from B to I, and a unique antiepimorphism ψ from B to J, such that $A=I\oplus J$ and $\theta=\phi+\psi$.*

PROOF. Let Π be the set of primitive ideals in A with the Jacobson topology. For each P in Π, let θ_p denote the product of θ and the natural quotient map from A onto A/P. Let $\Psi=\{P\in\Pi:\theta_p$ is a homomorphism$\}$ and $\Phi=\{P\in\Pi:\theta_p$ is an antihomomorphism$\}$. Herstein proves [3] (see also [10]) that a Jordan epimorphism to a prime ring is either a homomorphism or an antihomomorphism. Thus $\Psi\cup\Phi=\Pi$. Since each irreducible module has dimension greater than one, for each primitive ideal P in A there are x and y in B such that $\theta_p(x)$ and $\theta_p(y)$ do not commute. This shows that $\Psi\cap\Phi=\varnothing$. Let $J=\bigcap\{P\in\Psi\}$ and $I=\bigcap\{P\in\Phi\}$. Then the product of θ and the natural quotient map from A to A/J is a homomorphism, and the product of θ and the natural quotient map from A to A/I is an antihomomorphism. If P is a primitive ideal in A containing J, then θ_p is a homomorphism and so P is in Ψ. Hence Ψ is closed. Similarly Φ is closed. By the remarks before this theorem we obtain $A=I\oplus J$. Let ϕ denote the map

$$B\xrightarrow{\theta} A\rightarrow A/J\rightarrow I$$

and ψ denote the corresponding map with I and J interchanged, where $A/J\rightarrow I$ and $A/I\rightarrow J$ are the natural isomorphisms induced by the decomposition $A=I\oplus J$. Then ϕ is a homomorphism and ψ is an antihomomorphism.

Let $A=K\oplus L$ be a decomposition of A into ideals, ϕ' be an epimorphism from B to K, and ψ' be an antiepimorphism from B to L such that $\theta=\phi'+\psi'$. Let P be a primitive ideal in A such that θ_p is a homomorphism. If L is not contained in P, then an irreducible A module with kernel P is an irreducible L module with the same centralizer [5]. Hence there are x and y in $\theta^{-1}(L)$ such that $\theta_p(x)$ and $\theta_p(y)$ do not commute. By definition of L and ψ', $\theta|\theta^{-1}(L)$ is equal to ψ' so that $\theta_p|\theta^{-1}(L)$ is an antihomomorphism which contradicts the assumption that θ_p is a homomorphism. Hence L is contained in P, and so in J. Similarly K is contained in I. Because $I \oplus J = A = K \oplus L$, $L \subseteq J$, and $K \subseteq I$, we obtain $L = J$ and $K = I$. It now follows that $\phi' = \phi$ and $\psi' = \psi$.

2.2. THEOREM. *Let A and B be C^*-algebras, and let K and L be the minimal closed ideals in A and B, respectively, such that A/K and B/L are commutative. Let θ be a Jordan isomorphism from B to A. Then $\theta(L) = K$, and $\theta | L$ is the sum of a unique isomorphism and antiisomorphism.*

PROOF. We observe that K is the intersection of the kernels of all the characters on A. Let ω be a character on A. By Herstein's Theorem [**3**], $\omega\theta$ is a homomorphism or an antihomomorphism, so that $\omega\theta$ is a homomorphism. Since $\operatorname{Ker} \omega\theta = \theta^{-1} \operatorname{Ker} \omega \supseteq L$ for each character ω on A, $\theta^{-1}(K)$ contains L. Symmetry yields the reverse inclusion, so that $\theta(L)$ is equal to K.

We shall now consider θ as a Jordan isomorphism from the C^*-algebra L to the C^*-algebra K. Suppose that there is a character on K, that is, that K has a nondegenerate representation on one-dimensional Hilbert space. This representation extends to A, since K is a closed ideal in A [**2**, p. 52, 2.10.4], and gives rise to a character on A. However, a character on A annihilates K. This proves that K (and L) have no irreducible representations of dimension one.

In Theorem 2.1 the identity in the ring A (which corresponds to K in this theorem) was used only to show that the disjoint ideals I and J defined in Theorem 2.1 satisfy $I + J = A$. We prove that $I + J = K$, and note that the remainder of the proof is as in Theorem 2.1. Since $I \cap J = \{0\}$, $I + J$ is a closed ideal in K [**2**, p. 20, 1.9.12(b)] that is contained in no primitive ideal in K. For if P is a primitive ideal in K containing $I + J$, then θ_p is both a homomorphism and an antihomomorphism, since P contains I and J, which gives a contradiction as the representation of K with kernel P has dimension greater than one. In a C^*-algebra each closed ideal is the intersection of the primitive ideals in the algebra containing the closed ideal [**2**, p. 49, 2.9.7]. Therefore $I + J = K$.

2.3. EXAMPLE. If the condition that A has no irreducible modules of dimension one is omitted from Theorem 2.1, we might expect to find ideals I and J such that $I + J = A$, and I and J satisfy the other conclusions of Theorem 2.1 with the exception of the uniqueness of the decomposition. That this cannot be done is shown by the following example.

Let C be the algebra of continuous complex valued functions on $[0, 2]$, and let D be the ideal in C of functions that vanish at 1. Let

$$A = \begin{pmatrix} C & D \\ D & C \end{pmatrix}.$$

We define θ on A by

$$\theta(f_{ij}) = \begin{pmatrix} f_{11} & g \\ h & f_{22} \end{pmatrix}$$

where

$$\begin{aligned} g(x) &= f_{12}(x) \qquad \text{if } x \in [0, 1), \\ &= f_{21}(x) \qquad \text{if } x \in [1, 2], \end{aligned}$$

and h is defined similarly. Then θ is a Jordan isomorphism on A. There are no ideals I and J in A such that $I+J=A$ and there is a homomorphism ϕ from A into I and an antihomomorphism ψ from B into J with $\theta=\phi+\psi$.

3. **Continuous Jordan derivations on Banach algebras.** A linear map D on an algebra A is called a Jordan derivation if $D(a \circ b) = D(a) \circ b + a \circ D(b)$ for all a and b in A.

3.1. LEMMA. *Let D be a Jordan derivation on an algebra A, let P be a Jordan ideal in A, and let x be in P. Then $D^n(x^n)+P=n!D(x)^n+P$ for $n=1, 2, \cdots$.*

We omitt the proof of Lemma 3.1 as it uses the Leibnitz formula for D and the Jordan product in the same way that the proof of Lemma 2.1 of [9] uses the Leibnitz formula for an (associative) derivation.

3.2. LEMMA. *A continuous Jordan derivation on a Banach algebra leaves invariant the primitive ideals in the algebra.*

PROOF. Let P be a primitive ideal in a Banach algebra A, and let A have an irreducible representation on a Banach space X with kernel P. By Lemma 3.1, for each x in P, we have

$$\left\| \{D(x) + P\}^n \right\|^{1/n} = (n!)^{-1/n} \left\| D^n(x^n) + P \right\|^{1/n} \leqq (n!)^{-1/n} \cdot \|D\| \cdot \|x\|,$$

so that $D(x)+P$ is topologically nilpotent in A/P.

Let x be in P and suppose $D(x)$ is not in P. Then there are ξ and η in X such that $D(x)\xi=\eta$ and η is nonzero. Since $D(x)+P$ is topologically nilpotent, ξ and η are linearly independent over the centralizer of A on X as operators in the centralizer of A on X are continuous. Now we choose a y in A so that $y\xi=\xi$ and $y\eta=\xi-\eta$ [5]. Then $D(x) \circ y\xi = D(x)y\xi + yD(x)\xi = \xi$, so that $D(x) \circ y + P$ is not topologically nilpotent. However $D(x \circ y)+P=D(x) \circ y+x \circ D(y)+P$ $=D(x) \circ y+P$ is topologically nilpotent. This contradiction shows that $D(x)$ is in P and completes the proof.

3.3. THEOREM. *A continuous Jordan derivation on a semisimple Banach algebra is a derivation.*

PROOF. A continuous Jordan derivation on a Banach algebra leaves the primitive ideals in the algebra invariant by Lemma 3.2, and so may be dropped to a Jordan derivation on the algebra quotiented by a primitive ideal. From a theorem of Herstein [4] it follows that a Jordan derivation on a primitive Banach algebra is a derivation. This and the semisimplicity of the Banach algebra show that the Jordan derivation is a derivation.

3.4. QUESTION. Is a Jordan derivation on a semisimple Banach algebra continuous?

3.5. REMARK. A continuous linear operator T on a Banach space X over the complex field is said to be hermitian if exp itT is an isometry for each real t [12]. We shall use the result that T is hermitian if, and only if,

$$\{f(T): f(I) = \|f\| = 1, \quad f \in \mathcal{B}(X)^*\}$$

is contained in the real line, where $\mathcal{B}(X)$ is the Banach algebra of bounded linear operators on X [8]. We show that a hermitian operator *on* a C^*-algebra with identity is the sum of a left multiplication by a hermitian element in the algebra and a *-derivation.

Let A be a C^*-algebra with identity, and let T be a hermitian operator on A. For each continuous linear functional f on A of norm 1 with $f(1)=1$ (that is, each state on A), $B \rightarrow f(B(1))$ is a continuous linear functional on $\mathcal{B}(A)$ of norm 1 and equal to 1 at I. Thus $f(T(1))$ is contained in the real line for each state f on the C^*-algebra A, so that $T(1)$ is a hermitian element in A. The operator defined on A by left multiplication by a hermitian element in A is a hermitian operator on A. This follows from the definition of a hermitian operator and the result that the exponential of i times a hermitian element is a unitary element in a C^*-algebra. By subtracting the operator of left multiplication by $T(1)$ from T, we may assume that $T(1)$ is zero (the difference of two hermitian operators is hermitian [12, p. 122, Hilfssatz, Lemma 2]). Hence for all real t, exp itT is an isometry on A that maps 1 to 1. By a theorem of R. V. Kadison [7, p. 330, Theorem 7], it follows that exp itT is a Jordan *-isomorphism for all real t. Using this and by considering the derivative of exp itT at zero operating on a product, we find that T is a Jordan *-derivation on A (this is, a Jordan derivation on A such that $T(x^*) = -(Tx)^*$ for all x in A). Theorem 3.3 shows that T is a derivation, and this completes the proof.

Each *-derivation on a C^*-algebra is a hermitian operator on the C^*-algebra. This follows from the well-known results that the exponential of i times a *-derivation is a *-automorphism and that a *-automorphism on a C^*-algebra is an isometry [**7**].

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