

ON EMBEDDING OF LATTICES BELONGING TO THE SAME GENUS

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ABSTRACT. If R is an order in a semisimple algebra over a Dedekind ring and M, N two R -lattices in the same genus, an upper bound for the length of the composition series of M/N' for $N' \cong N$, is given. This answers a question posed by Rořter.

Let \mathfrak{o} be a Dedekind-ring whose quotient field k is an algebraic number field, A a semisimple algebra over k , and R an \mathfrak{o} -order in A . Two R -lattices M, N belong to the same genus Γ if $M_p \cong N_p$ for all primes p in \mathfrak{o} . In [2] Rořter posed the question whether every $X \in \Gamma$ is isomorphic to a maximal sublattice of M . The theorem below answers this question to the affirmative if A is simple, to the negative otherwise.

We will use notations and results from Jacobinski [1], which will be quoted as GD. Let M and N be in the same genus and $N \subset M$. We denote by $l_R(M/N)$ the length of a composition series of M/N as R -module. Clearly N is a maximal sublattice if and only if $l(M/N) = 1$. (For the definition of \mathfrak{L}'_R see GD, Definition 1.3, p. 5.)

THEOREM. *Let \mathfrak{o} be a Dedekind ring whose quotient field k is an algebraic number field and R an \mathfrak{o} -order in the semisimple k -algebra $A = \bigoplus A_i$, with A_i simple. Let M be an R -lattice in \mathfrak{L}'_R and let t_M be the number of the algebras A_i for which $A_i \otimes_{\mathfrak{o}} M \neq 0$. Then every lattice in the genus $\Gamma(M)$ is isomorphic to a lattice $N \subset M$ such that*

$$l_R(M/N) \leq t_M.$$

Moreover N can be chosen such that the annihilator of M/N is prime to an ideal d in \mathfrak{o} , given in advance.

PROOF. Let $U \neq \emptyset$ be a finite set of primes containing all p such that R_p is not a maximal order and also all primes dividing the given ideal d (see GD, p. 11). We embed R in a maximal order \mathfrak{D} and choose a two-sided \mathfrak{D} -ideal \mathfrak{F} , contained in R . For convenience we suppose that $\mathfrak{F}_p \neq \mathfrak{D}_p$ if and only if $p \in U$. As in GD, let $E(M), E(\mathfrak{D}M)$ denote the endomorphism-rings of M and $\mathfrak{D}M$ respectively.

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We replace Γ by the subset S of all $N \subset M$, such that the annihilator of M/N is not divisible by any prime of U . Every element of Γ is isomorphic to some $N \in S$, (GD, Proposition 2.1) and we have to find an $N \in S$ such that $l_R(M/N) \leq t_M$. Let \mathfrak{a} be an integral left $E(\mathfrak{D}M)$ -ideal such that $\mathfrak{a}_p = (1)$ for all $p \in U$. Then $M_{\mathfrak{a}} = M \cap \mathfrak{D}M_{\mathfrak{a}}$ is in S , and conversely, every element N of S determines a unique ideal \mathfrak{a} such that $N = M_{\mathfrak{a}}$ (GD, Proposition 21). This means that

$$\phi: \mathfrak{a} \rightarrow M \cap \mathfrak{D}M_{\mathfrak{a}}$$

is a 1-1 correspondence between integral $E(\mathfrak{D}M)$ -ideals with $\mathfrak{a}_p = (1)$, $p \in U$ and the elements of S . Since ϕ also preserves inclusions we have

$$l_R(M/N) = l_{E(\mathfrak{D}M)}(E(\mathfrak{D}M)/\mathfrak{a}).$$

The reduced norm $n(\mathfrak{a})$ is an integral ideal in $e_M C$, the center of $E(\mathfrak{D}M)$ (see GD, p. 4). Clearly $n(\mathfrak{a})$ is not divisible by any $p \in U$; moreover every such ideal in $e_M C$ is obtained as $n(\mathfrak{a})$, with $\mathfrak{a}_p = (1)$ for all $p \in U$. Now the multiplicativity of the reduced norm implies that

$$l_{E(\mathfrak{D}M)}(E(\mathfrak{D}M)/\mathfrak{a}) = l_{e_M C}(e_M C/n(\mathfrak{a})).$$

If we replace \mathfrak{a} by an ideal \mathfrak{b} , such that $n(\mathfrak{b}) \in n(\mathfrak{a})S_{\mathfrak{F}}(e_M)$, then the corresponding lattices N and V are isomorphic (GD, Lemma 2.6 and Theorem 2.2).

Let e_i denote the primitive central idempotents in A . Then we have

$$n(\mathfrak{a})S_{\mathfrak{F}}(e_M) = \bigoplus_{e_i M \neq 0} n(e_i \mathfrak{a}) \cdot S_{\mathfrak{F}}(e_i).$$

According to the generalized version of Dirichlet's theorem on arithmetic progressions, we can find a prime ideal p_i in each $n(e_i \mathfrak{a})S_{\mathfrak{F}}(e_i)$. If then we choose \mathfrak{b} such that

$$n(\mathfrak{b}) = \bigoplus_{e_i M \neq 0} p_i,$$

the corresponding lattice V is isomorphic to N and

$$l_R(M/V) = l_{e_M C}(e_M C/\mathfrak{b}) = t_M,$$

which completes the proof.

We now turn to the question whether the inequality in the theorem can be improved. For a particular genus Γ with $S_{\mathfrak{F}}(e_{\Gamma}) \neq H_{\Gamma}$, one sees from the proof that this may easily be the case. Moreover we have taken into account only lattices $N \subset M$ such that the annihilator of M/N is prime to \mathfrak{F} . Nevertheless the bound given is best possible, if

no special assumptions are made about the order R or the genus Γ . To see this choose A such that every maximal order $e_i\mathfrak{D}$ has class number > 1 ; for this it is sufficient that all $e_i\mathfrak{C}$ have class number > 1 .

Let e be a central idempotent in A and put $M = \mathfrak{D}e$. Then the genus $\Gamma(M)$ consists of all full fractionary ideals \mathfrak{A} in $\mathfrak{D}e$. Now choose an integral ideal $\mathfrak{A} \subset \mathfrak{D}e$, such that no $e_i\mathfrak{A}$ is principal for $e_i\mathfrak{A} \neq 0$. If $\mathfrak{B} \cong \mathfrak{A}$, then each $e_i\mathfrak{B} \neq e_i\mathfrak{D}$ since the $e_i\mathfrak{B}$ are not even principal. This implies that $l_{\mathfrak{D}}(M/\mathfrak{B}) \geq t_M$ for every $\mathfrak{B} \cong \mathfrak{A}$. Thus the constant t_M cannot in general be improved.

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