ON EMBEDDING OF LATTICES BELONGING TO
THE SAME GENUS

H. JACOBINSKI

Abstract. If $R$ is an order in a semisimple algebra over a
Dedekind ring and $M, N$ two $R$-lattices in the same genus, an up-
per bound for the length of the composition series of $M/N'$ for
$N'\cong N$, is given. This answers a question posed by Roiter.

Let $\mathfrak{o}$ be a Dedekind-ring whose quotient field $k$ is an algebraic
number field, $A$ a semisimple algebra over $k$, and $R$ an $\mathfrak{o}$-order in $A$. Two $R$-lattices $M, N$ belong to the same genus $\Gamma$ if $M_p=N_p$ for all
primes $p$ in $\mathfrak{o}$. In [2] Roiter posed the question whether every $X\in\Gamma$ is isomorphic to a maximal sublattice of $M$. The theorem below
answers this question to the affirmative if $A$ is simple, to the negative
otherwise.

We will use notations and results from Jacobinski [1], which will
be quoted as GD. Let $M$ and $N$ be in the same genus and $N\subseteq M$. We
denote by $l_R(M/N)$ the length of a composition series of $M/N$ as
$R$-module. Clearly $N$ is a maximal sublattice if and only if $l(M/N)
=1$. (For the definition of $l_R$ see GD, Definition 1.3, p. 5.)

Theorem. Let $\mathfrak{o}$ be a Dedekind ring whose quotient field $k$ is an alge-
braic number field and $R$ an $\mathfrak{o}$-order in the semisimple $k$-algebra $A =
\bigoplus A_i$, with $A_i$ simple. Let $M$ be an $R$-lattice in $\mathfrak{o}_k$ and let $t_M$ be the
number of the algebras $A_i$ for which $A_i\otimes_{\mathfrak{o}} M \neq 0$. Then every lattice in
the genus $\Gamma(M)$ is isomorphic to a lattice $N\subseteq M$ such that

$l_R(M/N) \leq t_M$.

Moreover $N$ can be chosen such that the annihilator of $M/N$ is prime to
an ideal $d$ in $\mathfrak{o}$, given in advance.

Proof. Let $U \neq \emptyset$ be a finite set of primes containing all $p$ such
that $R_p$ is not a maximal order and also all primes dividing the given
ideal $d$ (see GD, p. 11). We embed $R$ in a maximal order $\mathfrak{O}$ and choose
a two-sided $\mathfrak{O}$-ideal $\mathfrak{F}$, contained in $R$. For convenience we suppose
that $\mathfrak{F}_p \neq \mathfrak{O}_p$ if and only if $p \in U$. As in GD, let $E(M), E(\mathfrak{O}M)$ denote
the endomorphism-rings of $M$ and $\mathfrak{O}M$ respectively.
We replace $\Gamma$ by the subset $S$ of all $N \subset M$, such that the annihilator of $M/N$ is not divisible by any prime of $U$. Every element of $\Gamma$ is isomorphic to some $N \in S$, (GD, Proposition 2.1) and we have to find an $N \in S$ such that $l_R(M/N) \leq t_M$. Let $a$ be an integral left $E(\mathfrak{O}M)$-ideal such that $a_p = (1)$ for all $p \in U$. Then $M_a = M \cap \mathfrak{O}Ma$ is in $S$, and conversely, every element $N$ of $S$ determines a unique ideal $a$ such that $N = M_a$ (GD, Proposition 21). This means that

$$\phi: a \rightarrow M \cap \mathfrak{O}Ma$$

is a 1-1 correspondence between integral $E(\mathfrak{O}M)$-ideals with $a_p = (1)$, $p \in U$ and the elements of $S$. Since $\phi$ also preserves inclusions we have

$$l_R(M/N) = l_E(\mathfrak{O}M)(E(\mathfrak{O}M)/a).$$

The reduced norm $n(a)$ is an integral ideal in $e_MC$, the center of $E(\mathfrak{O}M)$ (see GD, p. 4). Clearly $n(a)$ is not divisible by any $p \in U$; moreover every such ideal in $e_MC$ is obtained as $n(a)$, with $a_p = (1)$ for all $p \in U$. Now the multiplicativity of the reduced norm implies that

$$l_E(\mathfrak{O}M)(E(\mathfrak{O}M)/a) = l_{e_M}(e_MC/n(a)).$$

If we replace $a$ by an ideal $b$, such that $n(b) \in n(a)S_{\mathfrak{F}}(e_M)$, then the corresponding lattices $N$ and $V$ are isomorphic (GD, Lemma 2.6 and Theorem 2.2).

Let $e_i$ denote the primitive central idempotents in $A$. Then we have

$$n(a)S_{\mathfrak{F}}(e_M) = \bigoplus_{e_i, M \neq 0} n(e_i a) \cdot S_{\mathfrak{F}}(e_i).$$

According to the generalized version of Dirichlet’s theorem on arithmetic progressions, we can find a prime ideal $p_i$ in each $n(e_i a)S_{\mathfrak{F}}(e_i)$. If then we choose $b$ such that

$$n(b) = \bigoplus_{e_i, M \neq 0} p_i,$$

the corresponding lattice $V$ is isomorphic to $N$ and

$$l_R(M/V) = l_{e_M}(e_MC/b) = t_M,$$

which completes the proof.

We now turn to the question whether the inequality in the theorem can be improved. For a particular genus $\Gamma$ with $S_{\mathfrak{F}}(e_\Gamma) \neq H_\Gamma$, one sees from the proof that this may easily be the case. Moreover we have taken into account only lattices $N \subset M$ such that the annihilator of $M/N$ is prime to $\mathfrak{F}$. Nevertheless the bound given is best possible, if
no special assumptions are made about the order $R$ or the genus $\Gamma$. To see this choose $A$ such that every maximal order $e_i\mathcal{O}$ has class number $>1$; for this it is sufficient that all $e_iC$ have class number $>1$.

Let $e$ be a central idempotent in $A$ and put $M = \mathcal{O}e$. Then the genus $\Gamma(M)$ consists of all full fractionary ideals $\mathfrak{A}$ in $\mathcal{O}e$. Now choose an integral ideal $\mathfrak{A}\subseteq\mathcal{O}e$, such that no $e_i\mathfrak{A}$ is principal for $e_i\mathfrak{A}\neq 0$. If $\mathfrak{B}\subseteq\mathfrak{A}$, then each $e_i\mathfrak{B}\neq e_i\mathcal{O}$ since the $e_i\mathfrak{B}$ are not even principal. This implies that $l_\mathfrak{O}(M/\mathfrak{B}) \geq t_M$ for every $\mathfrak{B}\subseteq\mathfrak{A}$. Thus the constant $t_M$ cannot in general be improved.

References


Chalmers University of Technology, Göteborg and University of Illinois