ON EMBEDDING OF LATTICES BELONGING TO THE SAME GENUS

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ABSTRACT. If R is an order in a semisimple algebra over a Dedekind ring and M, N two R-lattices in the same genus, an upper bound for the length of the composition series of M/N' for $N' \cong N$, is given. This answers a question posed by Roĭter.

Let $\mathfrak{o}$ be a Dedekind-ring whose quotient field k is an algebraic number field, A a semisimple algebra over k, and R an $\mathfrak{o}$-order in A. Two R-lattices M, N belong to the same genus Γ if $M_p \cong N_p$ for all primes p in $\mathfrak{o}$. In [2] Roĭter posed the question whether every $X \in \Gamma$ is isomorphic to a maximal sublattice of M. The theorem below answers this question to the affirmative if A is simple, to the negative otherwise.

We will use notations and results from Jacobinski [1], which will be quoted as GD. Let M and N be in the same genus and $N \subset M$. We denote by $l_R(M/N)$ the length of a composition series of M/N as R-module. Clearly N is a maximal sublattice if and only if $l(M/N) = 1$. (For the definition of $\mathfrak{L}'_R$ see GD, Definition 1.3, p. 5.)

THEOREM. *Let $\mathfrak{o}$ be a Dedekind ring whose quotient field k is an algebraic number field and R an $\mathfrak{o}$-order in the semisimple k-algebra $A = \oplus A_i$, with A_i simple. Let M be an R-lattice in $\mathfrak{L}'_R$ and let t_M be the number of the algebras A_i for which $A_i \otimes_{\mathfrak{o}} M \neq 0$. Then every lattice in the genus $\Gamma(M)$ is isomorphic to a lattice $N \subset M$ such that*

$$l_R(M/N) \leqq t_M.$$

Moreover N can be chosen such that the annihilator of M/N is prime to an ideal d in $\mathfrak{o}$, given in advance.

PROOF. Let $U \neq \emptyset$ be a finite set of primes containing all p such that R_p is not a maximal order and also all primes dividing the given ideal d (see GD, p. 11). We embed R in a maximal order $\mathfrak{O}$ and choose a two-sided $\mathfrak{O}$-ideal $\mathfrak{F}$, contained in R. For convenience we suppose that $\mathfrak{F}_p \neq \mathfrak{O}_p$ if and only if $p \in U$. As in GD, let $E(M)$, $E(\mathfrak{O}M)$ denote the endomorphism-rings of M and $\mathfrak{O}M$ respectively.

Received by the editors July 31, 1969.

AMS Subject Classifications. Primary 1640, 1075; Secondary 2080.

Key Phrases. Representation of orders over a Dedekind ring, genus of representation modules, isomorphism classes in a genus, Dirichlet's theorem on arithmetic progressions.

We replace Γ by the subset S of all $N \subset M$, such that the annihilator of M/N is not divisible by any prime of U. Every element of Γ is isomorphic to some $N \in S$, (GD, Proposition 2.1) and we have to find an $N \in S$ such that $l_R(M/N) \leqq t_M$. Let $\mathfrak{a}$ be an integral left $E(\mathfrak{O}M)$-ideal such that $\mathfrak{a}_p = (1)$ for all $p \in U$. Then $M_{\mathfrak{a}} = M \cap \mathfrak{O}M\mathfrak{a}$ is in S, and conversely, every element N of S determines a unique ideal $\mathfrak{a}$ such that $N = M_{\mathfrak{a}}$ (GD, Proposition 21). This means that

$$\phi: \mathfrak{a} \to M \cap \mathfrak{O}M_{\mathfrak{a}}$$

is a 1-1 correspondence between integral $E(\mathfrak{O}M)$-ideals with $\mathfrak{a}_p = (1)$, $p \in U$ and the elements of S. Since ϕ also preserves inclusions we have

$$l_R(M/N) = l_{E(OM)}(E(\mathfrak{O}M)/\mathfrak{a}).$$

The reduced norm $n(\mathfrak{a})$ is an integral ideal in $e_M C$, the center of $E(\mathfrak{O}M)$ (see GD, p. 4). Clearly $n(\mathfrak{a})$ is not divisible by any $p \in U$; moreover every such ideal in $e_M C$ is obtained as $n(\mathfrak{a})$, with $\mathfrak{a}_p = (1)$ for all $p \in U$. Now the multiplicativity of the reduced norm implies that

$$l_{E(\mathfrak{O}M)}(E(\mathfrak{O}M)/\mathfrak{a}) = l_{e_M C}(e_M C/n(\mathfrak{a})).$$

If we replace $\mathfrak{a}$ by an ideal $\mathfrak{b}$, such that $n(\mathfrak{b}) \in n(\mathfrak{a}) S_{\mathfrak{F}}(e_M)$, then the corresponding lattices N and V are isomorphic (GD, Lemma 2.6 and Theorem 2.2).

Let e_i denote the primitive central idempotents in A. Then we have

$$n(\mathfrak{a}) S_{\mathfrak{F}}(e_M) = \bigoplus_{e_i M \neq 0} n(e_i \mathfrak{a}) \cdot S_{\mathfrak{F}}(e_i).$$

According to the generalized version of Dirichlet's theorem on arithmetic progressions, we can find a prime ideal p_i in each $n(e_i\mathfrak{a}) S_{\mathfrak{F}}(e_i)$. If then we choose $\mathfrak{b}$ such that

$$n(\mathfrak{b}) = \bigoplus_{e_i M \neq 0} p_i,$$

the corresponding lattice V is isomorphic to N and

$$l_R(M/V) = l_{e_M C}(e_M C/\mathfrak{b}) = t_M,$$

which completes the proof.

We now turn to the question whether the inequality in the theorem can be improved. For a particular genus Γ with $S_{\mathfrak{F}}(e_\Gamma) \neq H_\Gamma$, one sees from the proof that this may easily be the case. Moreover we have taken into account only lattices $N \subset M$ such that the annihilator of M/N is prime to $\mathfrak{F}$. Nevertheless the bound given is best possible, if no special assumptions are made about the order R or the genus Γ. To see this choose A such that every maximal order $e_i\mathfrak{O}$ has class number >1; for this it is sufficient that all e_iC have class number >1.

Let e be a central idempotent in A and put $M=\mathfrak{O}e$. Then the genus $\Gamma(M)$ consists of all full fractionary ideals $\mathfrak{A}$ in $\mathfrak{O}e$. Now choose an integral ideal $\mathfrak{A}\subset\mathfrak{O}e$, such that no $e_i\mathfrak{A}$ is principal for $e_i\mathfrak{A}\neq 0$. If $\mathfrak{B}\cong\mathfrak{A}$, then each $e_i\mathfrak{B}\neq e_i\mathfrak{O}$ since the $e_i\mathfrak{B}$ are not even principal. This implies that $l_{\mathfrak{O}}(M/\mathfrak{B})\geqq t_M$ for every $\mathfrak{B}\cong\mathfrak{A}$. Thus the constant t_M cannot in general be improved.

References

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