

EQUIVALENT NORMS FOR SOBOLEV SPACES

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Let G be an open set in Euclidean n -space E_n , and let $C_0^\infty(G)$ be the (vector) space of infinitely differentiable functions with compact support in G . We consider two functionals from $C_0^\infty(G)$ to the real numbers defined by

$$(1) \quad |\phi|_{m,p,G} = \left\{ \sum_{|\alpha|=m} \int_G |D^\alpha \phi(x)|^p dx \right\}^{1/p},$$

$$(2) \quad \|\phi\|_{m,p,G} = \left\{ \sum_{j=0}^m |\phi|_{j,p,G}^p \right\}^{1/p}$$

where $1 \leq p < \infty$; $m = 0, 1, 2, \dots$; $\alpha = (\alpha_1, \dots, \alpha_n)$ is an n -tuple of nonnegative integers; $|\alpha| = \alpha_1 + \dots + \alpha_n$; $D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}$ where $D_j = \partial/\partial x_j$. The Sobolev space $W_0^{m,p}(G)$ is the Banach space obtained by completing $C_0^\infty(G)$ with respect to the norm (2). It is well known that if G is bounded then (1) is also a norm on $W_0^{m,p}(G)$ and it is equivalent to the usual norm (2). In particular if $p=2$ then the polyharmonic operator $(-\Delta)^m$ with null Dirichlet boundary data induces a positive definite operator in $W_0^{m,2}(G)$.

In this paper we show that (1) and (2) are equivalent norms on $W_0^{m,p}(G)$ for a suitable class of unbounded domains G having suitably regular $(n-1)$ -dimensional boundaries. In addition we strengthen somewhat a theorem on the compactness of certain imbeddings of Sobolev spaces on G which was obtained by the writer in [2].

We begin by noting that if (1) and (2) are equivalent norms in $W_0^{m,p}(G)$ then G must be *quasicylindrical*, i.e. $\text{dist.}(x, \text{bdry. } G)$ must remain bounded for x in G . Otherwise for $k=1, 2, \dots$ there would exist a ball $B_k \subset G$ having radius k , and a function u_k belonging to $C_0^\infty(B_k) \subset C_0^\infty(G)$ such that $u_k(x) = 1$ in a ball of radius $k-1$ and $|D^\alpha u_k(x)|$ is bounded independently of k and x . It follows that

$$(3) \quad |u_k|_{m,p,G}^p \leq \text{const. } k^{n-1}, \quad \|u_k\|_{m,p,G}^p \geq \text{const.}(k-1)^n$$

so that (1) and (2) cannot be equivalent norms on $W_0^{m,p}(G)$.

The property of being quasicylindrical is, however, not sufficient to guarantee the equivalence of (1) and (2). In fact if $p > 1$ and n

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$\geq mp$ and if G is a “pepper set” (i.e. if the boundary of G is a discrete set of points) we have shown [1, Theorem 1] that any $C_0^\infty(E_n)$ function can be modified so as to belong to $C_0^\infty(G)$ without increasing its $W^{m,p}$ norm or decreasing its L^p -norm by more than a specified amount ϵ . Hence we may construct functions u_k in $C_0^\infty(G)$ satisfying inequalities (3) and so (1) and (2) cannot be equivalent norms on $W_0^{m,p}(G)$.

In formulating a condition on G sufficient to guarantee the equivalence of norms (1) and (2) we require the following geometrical considerations. Let T be a solid torus in E_n obtained by rotating about an $(n-2)$ -dimensional subspace L of E_n an $(n-1)$ -dimensional ball B lying in a hyperplane H through L such that $B \cap L$ is empty. If (r, σ) denote polar coordinates in B with pole at the centre b of B we adjoin a coordinate θ representing the angle of rotation about L , to produce a system of toroidal coordinates (r, σ, θ) in T , with respect to which T is specified by the inequality $r \leq r_0$. The diameter of T is $2R$ where $R = r_0 + \text{dist.}(b, L)$. Let $\rho = \rho(r, \sigma)$ denote the distance of (r, σ, θ) from L so that $\rho \leq R$ in T . The volume element in the toroidal system is $dV = \rho d\theta dA$ where dA is the $(n-1)$ -volume element in B .

By a *slicing set* S for T we mean a set of points (r, σ, θ) containing exactly one point for each $(r, \sigma) \in B$.

LEMMA 1. *If S is a slicing set for T there exists a constant depending only on n and p and not on S such that*

$$\|\phi\|_{0,p,T} \leq \text{const. } R \|\phi\|_{1,p,T} \quad \text{for all } \phi \in C_0^\infty(E_n - S).$$

PROOF. S can be specified by an equation of the form $\theta = f(r, \sigma)$. Since ϕ vanishes in S we have

$$\phi(r, \sigma, \theta) = \int_{f(r,\sigma)}^\theta \frac{d}{dt} \phi(r, \sigma, t) dt.$$

Thus by Hölder’s inequality

$$|\phi(r, \sigma, \theta)|^p \leq (2\pi)^{p-1} \int_{f(r,\sigma)}^{f(r,\sigma)+2\pi} \left| \frac{d}{dt} \phi(r, \sigma, t) \right|^p dt.$$

Using the periodicity of ϕ in θ we now obtain

$$\begin{aligned} \|\phi\|_{0,p,T}^p &\leq (2\pi)^{p-1} \int_B \rho(r, \sigma) dA \int_0^{2\pi} d\theta \int_0^{2\pi} \left| \frac{d}{dt} \phi(r, \sigma, t) \right|^p dt \\ &\leq (2\pi)^p \left| \frac{d\phi}{d\theta} \right|_{0,p,T}^p \\ &\leq \text{const. } R^p \|\phi\|_{1,p,T}^p \end{aligned}$$

since $|d\phi/d\theta| \leq R \sum_{i=1}^n |\partial\phi/\partial x_i|$ in T .

DEFINITION. An open set G in E_n is said to satisfy condition (T) if there exists a cover of G by a family \mathfrak{J} of solid tori satisfying for some constant R and integer N

- (i) $\text{diam. } T \leq 2R$ for all $T \in \mathfrak{J}$,
- (ii) the intersection of any $N+1$ tori in \mathfrak{J} is empty,
- (iii) each $T \in \mathfrak{J}$ has a slicing set $S \subset \text{bdry. } G$.

Condition (T) certainly implies that G is quasicylindrical. Roughly speaking G will satisfy condition (T) if the $(n-1)$ -dimensional part of its boundary is sufficiently regular and bounds a quasicylindrical domain containing G . An example is the "spiny urchin" of [3].

THEOREM 1. *If G satisfies condition (T) then (1) and (2) are equivalent norms in $W_0^{m,p}(G)$.*

PROOF. By Lemma 1 there is a constant K independent of $T \in \mathfrak{J}$ such that $\|\phi\|_{0,p,T} \leq K \|\phi\|_{1,p,T}$ for all $\phi \in C_0^\infty(G)$. Since $G \subset \cup_{\mathfrak{J}} T$ and any $N+1$ of the tori T have empty intersection, we obtain

$$\|\phi\|_{0,p,G}^p \leq \sum_{T \in \mathfrak{J}} \|\phi\|_{0,p,T}^p \leq K^p \sum_{T \in \mathfrak{J}} \|\phi\|_{1,p,T}^p \leq K^p N \|\phi\|_{1,p,G}^p.$$

Application of this inequality to various derivatives $D^\alpha \phi$ yields

$$\|\phi\|_{j,p,G} \leq \text{const.} \|\phi\|_{j+1,p,G}, \quad j = 0, 1, 2, \dots$$

whence we obtain by iteration

$$\|\phi\|_{m,p,G}^p \leq \|\phi\|_{m,p,G}^p = \sum_{j=0}^m \|\phi\|_{j,p,G}^p \leq \text{const.} \|\phi\|_{m,p,G}^p.$$

The theorem then follows by completion.

COROLLARY. *The functionals $\|\cdot\|_{m,p,k,G}$ defined by*

$$\|u\|_{m,p,k,G}^p = \sum_{k \leq j \leq m} \|u\|_{j,p,G}^p$$

are equivalent norms on $W_0^{m,p}(G)$ for $k=0, 1, 2, \dots, m$ provided G satisfies condition (T) .

APPLICATION. The polyharmonic operator in G with null Dirichlet boundary data,

$$\begin{aligned} (-\Delta)^m u(x) &= (-1)^m (D_1^2 + \dots + D_n^2)^m u(x), & x \in G, \\ D^\alpha u(x) &= 0, & x \in \text{bdry } G, \quad |\alpha| \leq m-1 \end{aligned}$$

is positive definite in G if G satisfies condition (T). In fact $(-\Delta)^m$ has Dirichlet form

$$d(\phi, \psi) = \sum_{|\alpha|=m} \binom{m}{\alpha} \int_G D^\alpha \phi(x) D^\alpha \psi(x) dx$$

where $\phi, \psi \in C_0^\infty(G)$. Since

$$|d(\phi, \psi)| \leq \text{const.} \|\phi\|_{m,2,G} \|\psi\|_{m,2,G}$$

and since

$$\begin{aligned} d(\phi, \phi) &= \sum_{|\alpha|=m} \binom{m}{\alpha} \|\tilde{D}^\alpha \phi\|_{0,2,G}^2 \\ &\geq \text{const.} \|\phi\|_{m,2,G}^2 \geq \text{const.} \|\phi\|_{m,2,G}^2 \end{aligned}$$

it follows by the Lax-Milgram theorem that there exists a linear homeomorphism L of $W_0^{m,2}(G)$ onto itself such that $d(u, v) = \langle Lu, v \rangle_m$ where $\langle \cdot, \cdot \rangle_m$ is the inner product in the Hilbert space $W_0^{m,2}(G)$. Hence the generalized Dirichlet problem [4, p. 98] for the polyharmonic operator has a unique solution.

REMARK. In a recent paper [2] the writer has given a sufficient condition on an unbounded domain G to guarantee that the Sobolev space imbeddings

$$\begin{aligned} W_0^{m,p}(G) \rightarrow W_0^{j,r}(G), \quad m > j, \quad n \geq (m-j)p, \\ p \leq r < np(n - mp + jp)^{-1} \end{aligned}$$

are compact. A slight modification of condition (T) which insures that G is quasibounded ($\text{dist.}(x, \text{bdry } G) \rightarrow 0$ as $|x| \rightarrow \infty, x \in G$) instead of just quasicylindrical, yields a new sufficient condition for this compactness which is weaker than the condition of that paper. Specifically, in place of part (i) of condition (T) we should assume:

(i') $\text{diam } T \rightarrow 0$ as $\text{dist}(0, T) \rightarrow \infty$. Lemma 3 now guarantees $\|\phi\|_{0,p,T} \leq K(\text{diam } T) \|\phi\|_{m,p,T}$ where $K(\text{diam } T) \rightarrow 0$ as $\text{dist}(0, T) \rightarrow \infty$. The remainder of the compactness proof is similar to that in [2].

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