EQUIVALENT NORMS FOR SOBOLEV SPACES

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Let G be an open set in Euclidean n-space E_n, and let $C_0^\infty(G)$ be the (vector) space of infinitely differentiable functions with compact support in G. We consider two functionals from $C_0^\infty(G)$ to the real numbers defined by

$$(1) \qquad |\phi|_{m,p,G} = \left\{ \sum_{|\alpha|=m} \int_G |D^\alpha \phi(x)|^p dx \right\}^{1/p},$$

$$(2) \qquad \|\phi\|_{m,p,G} = \left\{ \sum_{j=0}^{m} |\phi|_{j,p,G}^p \right\}^{1/p}$$

where $1 \leqq p < \infty$; $m = 0, 1, 2, \cdots$; $\alpha = (\alpha_1, \cdots, \alpha_n)$ is an n-tuple of nonnegative integers; $|\alpha| = \alpha_1 + \cdots + \alpha_n$; $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$ where $D_j = \partial/\partial x_j$. The Sobolev space $W_0^{m,p}(G)$ is the Banach space obtained by completing $C_0^\infty(G)$ with respect to the norm (2). It is well known that if G is bounded then (1) is also a norm on $W_0^{m,p}(G)$ and it is equivalent to the usual norm (2). In particular if $p=2$ then the polyharmonic operator $(-\Delta)^m$ with null Dirichlet boundary data induces a positive definite operator in $W_0^{m,2}(G)$.

In this paper we show that (1) and (2) are equivalent norms on $W_0^{m,p}(G)$ for a suitable class of unbounded domains G having suitably regular $(n-1)$-dimensional boundaries. In addition we strengthen somewhat a theorem on the compactness of certain imbeddings of Sobolev spaces on G which was obtained by the writer in [2].

We begin by noting that if (1) and (2) are equivalent norms in $W_0^{m,p}(G)$ then G must be *quasicylindrical*, i.e. dist.$(x$, bdry. $G)$ must remain bounded for x in G. Otherwise for $k = 1, 2, \cdots$ there would exist a ball $B_k \subset G$ having radius k, and a function u_k belonging to $C_0^\infty(B_k) \subset C_0^\infty(G)$ such that $u_k(x) = 1$ in a ball of radius $k-1$ and $|D^\alpha u_k(x)|$ is bounded independently of k and x. It follows that

$$(3) \qquad |u_k|_{m,p,G}^p \leqq \text{const. } k^{n-1}, \qquad \|u_k\|_{m,p,G}^p \geqq \text{const.}(k-1)^n$$

so that (1) and (2) cannot be equivalent norms on $W_0^{m,p}(G)$.

The property of being quasicylindrical is, however, not sufficient to guarantee the equivalence of (1) and (2). In fact if $p>1$ and n

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$\geqq mp$ and if G is a "pepper set" (i.e. if the boundary of G is a discrete set of points) we have shown [**1**, Theorem 1] that any $C_0^\infty(E_n)$ function can be modified so as to belong to $C_0^\infty(G)$ without increasing its $W^{m,p}$ norm or decreasing its L^p-norm by more than a specified amount ϵ. Hence we may construct functions u_k in $C_0^\infty(G)$ satisfying inequalities (3) and so (1) and (2) cannot be equivalent norms on $W_0^{m,p}(G)$.

In formulating a condition on G sufficient to guarantee the equivalence of norms (1) and (2) we require the following geometrical considerations. Let T be a solid torus in E_n obtained by rotating about an $(n-2)$-dimensional subspace L of E_n an $(n-1)$-dimensional ball B lying in a hyperplane H through L such that $B\cap L$ is empty. If (r, σ) denote polar coordinates in B with pole at the centre b of B we adjoin a coordinate θ representing the angle of rotation about L, to produce a system of toroidal coordinates (r, σ, θ) in T, with respect to which T is specified by the inequality $r\leqq r_0$. The diameter of T is $2R$ where $R=r_0+\text{dist.}(b, L)$. Let $\rho=\rho(r, \sigma)$ denote the distance of (r, σ, θ) from L so that $\rho\leqq R$ in T. The volume element in the toroidal system is $dV=\rho d\theta dA$ where dA is the $(n-1)$-volume element in B.

By a *slicing set* S for T we mean a set of points (r, σ, θ) containing exactly one point for each $(r, \sigma)\in B$.

LEMMA 1. *If S is a slicing set for T there exists a constant depending only on n and p and not on S such that*

$$\|\phi\|_{0,p,T} \leqq \text{const. } R\,|\phi|_{1,p,T} \qquad \text{for all } \phi\in C_0^\infty(E_n-S).$$

PROOF. S can be specified by an equation of the form $\theta=f(r, \sigma)$. Since ϕ vanishes in S we have

$$\phi(r, \sigma, \theta) = \int_{f(r,\sigma)}^{\theta} \frac{d}{dt}\phi(r, \sigma, t)dt.$$

Thus by Hölder's inequality

$$|\phi(r, \sigma, \theta)|^p \leqq (2\pi)^{p-1}\int_{f(r,\sigma)}^{f(r,\sigma)+2\pi}\left|\frac{d}{dt}\phi(r, \sigma, t)\right|^p dt.$$

Using the periodicity of ϕ in θ we now obtain

$$\begin{aligned}\|\phi\|_{0,p,T}^p &\leqq (2\pi)^{p-1}\int_B \rho(r, \sigma)dA\int_0^{2\pi}d\theta\int_0^{2\pi}\left|\frac{d}{dt}\phi(r, \sigma, t)\right|^p dt\\ &\leqq (2\pi)^p\left|\frac{d\phi}{d\theta}\right|_{0,p,T}^p\\ &\leqq \text{const. } R^p\,|\phi|_{1,p,T}^p\end{aligned}$$

since $|d\phi/d\theta| \leqq R\sum_{i=1}^{n} |\partial\phi/\partial x_i|$ in T.

DEFINITION. An open set G in E_n is said to satisfy condition (T) if there exists a cover of G by a family $\mathfrak{J}$ of solid tori satisfying for some constant R and integer N

(i) diam. $T \leqq 2R$ for all $T \in \mathfrak{J}$,

(ii) the intersection of any $N+1$ tori in $\mathfrak{J}$ is empty,

(iii) each $T \in \mathfrak{J}$ has a slicing set $S \subset$ bdry. G.

Condition (T) certainly implies that G is quasicylindrical. Roughly speaking G will satisfy condition (T) if the $(n-1)$-dimensional part of its boundary is sufficiently regular and bounds a quasicylindrical domain containing G. An example is the "spiny urchin" of [3].

THEOREM 1. *If G satisfies condition* (T) *then* (1) *and* (2) *are equivalent norms in* $W_0^{m,p}(G)$.

PROOF. By Lemma 1 there is a constant K independent of $T \in \mathfrak{J}$ such that $\|\phi\|_{0,p,T} \leqq K|\phi|_{1,p,T}$ for all $\phi \in C_0^\infty(G)$. Since $G \subset \bigcup_{\mathfrak{J}} T$ and any $N+1$ of the tori T have empty intersection, we obtain

$$\|\phi\|_{0,p,G}^{p} \leqq \sum_{T\in\mathfrak{J}} \|\phi\|_{0,p,T}^{p} \leqq K^{p} \sum_{T\in\mathfrak{J}} |\phi|_{1,p,T}^{p} \leqq K^{p} N |\phi|_{1,p,G}^{p}.$$

Application of this inequality to various derivatives $D^\alpha\phi$ yields

$$|\phi|_{j,p,G} \leqq \text{const.}\, |\phi|_{j+1,p,G}, \qquad j = 0, 1, 2, \cdots$$

whence we obtain by iteration

$$|\phi|_{m,p,G}^{p} \leqq \|\phi\|_{m,p,G}^{p} = \sum_{j=0}^{m} |\phi|_{j,p,G}^{p} \leqq \text{const.}\, |\phi|_{m,p,G}^{p}.$$

The theorem then follows by completion.

COROLLARY. *The functionals* $\|\cdot\|_{m,p,k,G}$ *defined by*

$$\|u\|_{m,p,k,G}^{p} = \sum_{k\leqq j\leqq m} |u|_{j,p,G}^{p}$$

are equivalent norms on $W_0^{m,p}(G)$ *for* $k=0, 1, 2, \cdots, m$ *provided G satisfies condition* (T).

APPLICATION. The polyharmonic operator in G with null Dirichlet boundary data,

$$(-\Delta)^m u(x) = (-1)^m (D_1^2 + \cdots + D_n^2)^m u(x), \qquad x \in G,$$
$$D^\alpha u(x) = 0, \qquad x \in \text{bdry}\, G, \qquad |\alpha| \leqq m-1$$

is positive definite in G if G satisfies condition (T). In fact $(-\Delta)^m$ has Dirichlet form

$$d(\phi, \psi) = \sum_{|\alpha|=m} \binom{m}{\alpha} \int_G D^\alpha \phi(x) D^\alpha \psi(x) dx$$

where $\phi, \psi \in C_0^\infty(G)$. Since

$$| d(\phi, \psi) | \leqq \text{const.} \|\phi\|_{m,2,G} \|\psi\|_{m,2,G}$$

and since

$$d(\phi, \phi) = \sum_{|\alpha|=m} \binom{m}{\alpha} \|\overset{\alpha}{D} \phi\|_{0,2,G}^2$$

$$\geqq \text{const.} | \phi |_{m,2,G}^2 \geqq \text{const.} \|\phi\|_{m,2,G}^2$$

it follows by the Lax-Milgram theorem that there exists a linear homeomorphism L of $W_0^{m,2}(G)$ onto itself such that $d(u, v) = \langle Lu, v\rangle_m$ where $\langle \cdot, \cdot\rangle_m$ is the inner product in the Hilbert space $W_0^{m,2}(G)$. Hence the generalized Dirichlet problem [4, p. 98] for the polyharmonic operator has a unique solution.

Remark. In a recent paper [2] the writer has given a sufficient condition on an unbounded domain G to guarantee that the Sobolev space imbeddings

$$W_0^{m,p}(G) \to W_0^{j,r}(G), \qquad m > j, \qquad n \geqq (m-j)p,$$
$$p \leqq r < np(n - mp + jp)^{-1}$$

are compact. A slight modification of condition (T) which insures that G is quasibounded (dist. $(x$, bdry $G) \to 0$ as $|x| \to \infty$, $x \in G$) instead of just quasicylindrical, yields a new sufficient condition for this compactness which is weaker than the condition of that paper. Specifically, in place of part (i) of condition (T) we should assume:

(i′) diam $T \to 0$ as dist $(0, T) \to \infty$. Lemma 3 now guarantees $\|\phi\|_{0,p,T} \leqq K(\text{diam } T) |\phi|_{m,p,T}$ where $K(\text{diam } T) \to 0$ as $\text{dist}(0, T) \to \infty$. The remainder of the compactness proof is similar to that in [2].

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