ON A THEOREM OF PREISSMANN

WILLIAM P. BYERS

In this note we strengthen a theorem of Preissman [3] which states that the only abelian subgroups of the fundamental group of a compact manifold of negative curvature are infinite cyclic. Our result is

THEOREM. *Let M be a compact Riemannian manifold with all sectional curvatures less than zero. If H is a solvable subgroup of $\pi_1(M)$ then H must be infinite cyclic. In particular $\pi_1(M)$ has no solvable subgroup of finite index.*

Let $\pi: \overline{M} \to M$ be a universal covering of M with the induced metric on $\overline{M}$. We identify the fundamental group $\pi_1(M)$ with the group of all covering transformations of this covering. We need the following (well-known) lemmas:

LEMMA 1. *Let $\alpha: \overline{M} \to \overline{M}$ be any covering transformation. Then α maps some geodesic of $\overline{M}$ into itself.*

PROOF. Let $x_0 \in M$ and $\bar{x}_0 \in \overline{M}$ such that $\pi(\bar{x}_0) = x_0$. Let $\overline{C}$ be a path in $\overline{M}$ joining $\bar{x}_0$ to $\alpha(\bar{x}_0)$ and let C be its projection into M. According to Bishop and Crittenden [1, p. 293] there is a closed geodesic in every free homotopy class of loops on M. Let C_1 be a closed geodesic which is free homotopic to C and let F: $[0, 1] \times [0, 1] \to M$ be the free homotopy joining C to C_1, i.e. $F(x, 0) = C(x)$, $F(x, 1) = C_1(x)$, $F(0, t) = F(1, t)$. Let $G(t) = F(0, t)$ and let $\overline{G}$ be the unique lift of this path to $\overline{M}$ such that $\overline{G}(0) = \bar{x}_0$. Let $\overline{G}(1) = \bar{y}_0$. It follows from elementary properties of covering spaces that the loop C_1 and the base point $\bar{y}_0$ induce the transformation α. However, the former transformation obviously preserves the geodesic in $\overline{M}$ which passes through $\bar{y}_0$ and covers C_1. Q.E.D.

LEMMA 2. *Each covering transformation preserves at most one geodesic of $\overline{M}$.*

PROOF. Let C_1 and C_2 be two geodesics preserved by $\alpha \in \pi_1(M)$. Then $C_1 \cap C_2 = \emptyset$ since α would fix any point of intersection. If $t \in \mathbf{R}$ is a parametrization by arc length along C_1 then (E. Cartan [2]) the function $f(t)$ which measures the distance to C_2 achieves no local maxima. Thus it must be either increasing, decreasing, or decreasing to a minimum and then increasing. However, the distance of $x \in C_1$

Received by the editors June 13, 1969.

to C_2 equals the distance from $\alpha^n(x)$ to C_2 since α is an isometry. Thus the function f reduces to a constant function which is impossible on a manifold of strictly negative curvature. Q.E.D.

PROOF OF THE THEOREM. Let $H=H_0\supset H_1\supset H_2\supset \cdots \supset H_{k-2} \supset H_{k-1}\supset H_k=\{1\}$ be the derived series for H. Let α, $\beta\in H_{k-1}$ which is abelian and suppose α preserves the geodesic C. Then $\alpha(\beta(C)) =\beta(\alpha(C))=\beta(C)$. From Lemma 2, $\beta(C)=C$. This implies that α and β are generated by the same closed geodesic on M. Suppose γ is the transformation which corresponds to going around this closed geodesic exactly once, with $\gamma(C)=C$. Then we have shown that $H_{k-1} \subseteq \{\gamma^n\}_{n\in Z}$. Thus H_{k-2} is infinite cyclic if $\{\gamma^n\}_{n\in Z}$ is. That $\{\gamma^n\}_{n\in Z}$ is infinite cyclic follows from the fact that $\overline{M}$ is diffeomorphic to Euclidean space and a result of P. A. Smith [4] stating that Z_p cannot act freely on R^n.

Let $\delta\in H_{k-2}$. Then $\delta^{-1}\alpha^{-1}\delta\alpha=\gamma^n$ for some $n\in Z$. Thus $\delta^{-1}\alpha^{-1}\delta\alpha(C) =\gamma^n(C)=C$, i.e. $\alpha^{-1}\delta(C)=\delta(C)$. Therefore α preserves $\delta(C)$ and $\delta(C)=C$ as above. Again this implies that $\delta=\gamma^m$ for some $m\in Z$ which shows that $H_{k-2}\subseteq\{\gamma^n\}$. Proceeding in this way we eventually show that $H\subseteq\{\gamma^n\}_{n\in Z}$ and therefore H is infinite cyclic.

To prove the final statement, let $\{\gamma^n\}_{n\in Z}$ be the solvable subgroup of finite index which we now know to be infinite cyclic. Suppose α is some transformation not in that subgroup. Then $\alpha^n=\gamma^m$ for some n, $m\in Z$ since the subgroup has finite index. Suppose $\gamma(C)=C$. Then $\alpha^n(C)=C$. However, since α preserves a unique geodesic we must have $\alpha(C)=C$. As above this implies that $\pi_1(M)$ is infinite cyclic. However, this is impossible for compact M [3, p. 206].

REFERENCES

1. R. L. Bishop and R. J. Crittenden, *Geometry of manifolds*, Pure and Appl. Math., vol. 15, Academic Press, New York, 1964. MR 29 #6401.

2. E. Cartan, *Leçons sur la géométrie des espaces de Riemann*, 2nd ed., Gauthier-Villars, Paris, 1951. MR 13, 491.

3. A. Preissmann, *Quelques propriétés globales des espaces de Riemann*, Comment. Math. Helv. 15 (1943), 175–216. MR 6, 20.

4. P. A. Smith, "Fixed points of periodic transformations," Appendix B in Lefschetz, *Algebraic topology*, Amer. Math. Soc. Colloq. Publ., vol. 27, Amer. Math. Soc., Providence, R. I., 1942. MR 4, 84.

UNIVERSITY OF CALIFORNIA, BERKELEY