PROJECTIVE REPRESENTATIONS AND INDUCED LINEAR CHARACTERS[1]

T. V. FOSSUM

0. Introduction and notation. The purpose of this paper is to generalize arithmetic results obtained by C. W. Curtis and the author in [1] concerning the degrees of irreducible representations of finite groups. Although the generalization given in §2 is somewhat crude because of the multitude of hypotheses, the necessary groundwork devoted to projective representations in §1 seems to be of interest in itself.

All groups considered here are finite. Let G be a group, H a subgroup of G, and suppose $x \in G$. We will adhere to the following notations throughout this paper:

$$|G| = \text{order of } G, \qquad h^x = x^{-1}hx \quad (h \in H),$$
$$H^x = \{h^x : h \in H\}, \qquad H^{(x)} = H \cap H^x, \qquad \operatorname{ind}_H x = [H : H^{(x)}].$$

The symbol K will always stand for the complex field, and all characters are assumed to be K-characters. If ζ is a character of H, define $\zeta^x: H^x \to K$ by $\zeta^x(h^x) = \zeta(h)$, $(h^x \in H^x)$. Then ζ^x is a K-character of H^x. All other notations are standard and are included in [2].

As a foundation for our results we state the following theorem, the proof of which is the subject of [1].

THEOREM A. *Let H be a subgroup of the group G. Suppose ζ is a linear character of H and χ an irreducible character of G such that $(\zeta^G, \chi) > 0$. Set $e = |H|^{-1} \sum_{h \in H} \zeta(h^{-1})h$, the primitive idempotent in KH such that KHe affords ζ. Then the restriction χ_E of χ to $E = eKGe$ is an irreducible character of E. If $\{g_i\}$ is a set of representatives of the distinct (H, H)-double cosets $H\chi H$ for which $\zeta^x = \zeta$ on $H^{(x)}$, then the set of $a_i = (\operatorname{ind}_H g_i)eg_ie$ is a basis for $E = eKGe$. Moreover if $b_i = (\operatorname{ind}_H g_i)eg_i^{-1}e$, then the block idempotent of E corresponding to χ_E is*

$$(\chi(1)/[G: H]) \sum_i (\operatorname{ind}_H g_i)^{-1}\chi(b_i)a_i.$$

Finally, the degree $\chi(1)$ of χ divides

$$[G: H] \operatorname{lcm}\{\operatorname{ind}_H g_i\} = [G: H] \operatorname{lcm}\{\operatorname{ind}_H x : \zeta^x = \zeta \text{ on } H^{(x)}, x \in G].$$

Received by the editors May 26, 1969.

[1] This paper is a portion of a Ph.D. thesis by the author submitted to the University of Oregon, August, 1968.

1. Characters and twisted group-algebras. Assume S is a group, $f: S\times S\to K^* = K\setminus\{0\}$ a factor set of S whose values are powers of a fixed primitive nth root of unity w in K. (Every factor set can be so normalized. See [2, §53].) Let $K(S)_f$ be the (associative) twisted group-algebra with K-basis $\{[s]: s\in S\}$ in one-to-one correspondence with the elements of S and with multiplication given by $[g][h] = f(g, h)[gh]$, $(g, h\in S)$. Finally assume $[1]$ is the multiplicative identity for $K(S)_f$.

Let S_f denote the finite multiplicative subgroup of the group of units of $K(S)_f$ consisting of elements of the form $w^i[s]$, $0\leqq i\leqq n-1$, $s\in S$. Then $[1]$ is the identity of S_f and S_f contains a cyclic central subgroup $C_f = \{w^i[1]: 0\leqq i\leqq n-1\}$ such that $S_f/C_f\simeq S$. If KS_f denotes the ordinary group-algebra of S_f over K then the map $\Phi: KS_f \to K(S)_f$ given by $\Phi(w^i[s]) = w^i\cdot[s]$ (extended linearly to KS_f) is a K-algebra epimorphism. The element

$$e_f = 1/n\sum_{i=0}^{n-1} w^{-i}(w^i[1])$$

is an idempotent in KC_f (see [2, §33]) which is central in KS_f such that $\Phi(e_f) = [1]$, and $e_f(KS_f)e_f\simeq K(S)_f$ under Φ.

Now suppose T is a subgroup of S and construct the unital subalgebra $K(T)_f$ of $K(S)_f$ with the factor set f restricted to $T\times T$. Similar to S_f construct $T_f = \{w^i[t]\}$ with cyclic central subgroup C_f such that $T_f/C_f\simeq T$. Then the restriction Φ' of Φ to KT_f is an algebra epimorphism of KT_f onto $K(T)_f$.

Suppose χ is an irreducible character of $K(S)_f$. We pull back χ to KS_f by defining $\chi_f = \chi\Phi$. Then χ_f is an irreducible character of KS_f of degree $\chi_f([1]) = \chi([1])$ and its corresponding block idempotent in KS_f is

$$\epsilon = \chi_f([1])\,|S_f|^{-1}\sum_{g\in S_f}\chi_f(g^{-1})g$$

(see [2, §33]). By definition of Φ,

$$\Phi(\epsilon) = \chi([1])\,|S|^{-1}\sum_{s\in S}\chi([s]^{-1})[s],$$

since $|S_f| = n|S|$ and $\chi([1]) = \chi_f([1])$. Moreover $\Phi(\epsilon)$ is the block idempotent in $K(S)_f$ corresponding to χ.

Suppose further that $\zeta: K(T)_f\to K$ is an algebra homomorphism.[2] Then we pull back ζ to $\zeta_f = \zeta\Phi'$, which is clearly an algebra homomorphism of KT_f onto K. If χ restricted to $K(T)_f$ contains ζ (i.e., if

[2] Since ζ is linear on $K(T)_f$, f is a trivial cocycle on T and therefore $K(T)_f\simeq KT$.

the module affording χ, restricted to $K(T)_f$, has the module affording ζ as a composition factor) then χ_f, restricted to KT_f, contains ζ_f. By Theorem A above, the degree $\chi_f([1])$ of χ_f divides

$$[S_f: T_f] \operatorname{lcm}\{\operatorname{ind}_{T_f} g: \zeta_f^g = \zeta_f \text{ on } T_f^{(g)}, g \in S_f\}.$$

First note that $[S_f: T_f] = [S: T]$. Moreover one checks that

$$\operatorname{lcm}\{\operatorname{ind}_{T_f} g: \zeta_f^g = \zeta_f \text{ on } T_f^{(g)}, g \in S_f\}$$
$$= \operatorname{lcm}\{\operatorname{ind}_T x: \zeta^{[x]} = \zeta \text{ on } K(T^{(x)})_f, x \in S\},$$

where $\zeta^{[x]}([t]) = \zeta([x][t][x]^{-1})$ for $t \in T^{(x)}$ defines $\zeta^{[x]}$ on $K(T^{(x)})_f$.

The above development proves the following analogue to the last part of Theorem A for twisted group algebras:

THEOREM B. *Let S be a group with factor set f and T a subgroup of S. Assume that f has values which are powers of a fixed primitive nth root of unity in K and that $[1]$ is the identity for $K(S)_f$. If χ is an irreducible character of $K(S)_f$ and ζ is a linear character of $K(T)_f$ such that ζ appears in the restriction of χ to $K(T)_f$, then the degree $\chi([1])$ of χ divides*

$$[S: T] \operatorname{lcm}\{\operatorname{ind}_T x: \zeta^{[x]} = \zeta \text{ on } K(T^{(x)})_f, x \in S\},$$

where $\zeta^{[x]}([t]) = \zeta([x][t][x]^{-1})$ for all $t \in T^{(x)}$.

2. **The two-step case.** Let G be a group, N a normal subgroup of G, and suppose H is a subgroup of G such that $N \subseteq H \subseteq G$. Let ψ, ζ, χ be irreducible characters of N, H, G, respectively, such that they satisfy the following compatibility conditions:

$$(*) \qquad \psi(1) = 1, \qquad (\psi^H, \zeta) = 1, \qquad (\zeta^G, \chi) \geqq 1.$$

Let e be the idempotent in KN such that KNe affords ψ, namely $e = |N|^{-1} \sum_{n \in N} \psi(n^{-1})n$, and let $E = eKGe$. Set $S = \{g \in G: \psi^g = \psi \text{ on } N\}$; observe that $N^{(g)} = N$ for all $g \in G$ since N is normal in G, and S is a subgroup of G containing N. Write $S = \bigcup g_i N$ (disjoint). By Theorem A, $E = eKGe$ has a basis consisting of $\{eg_ie\}$. Note that for $g \in S$, $ege = eg = ge$. Further, in writing $S = \bigcup g_i N$ (disjoint) we assume that $g_1 = 1$, and we fix this coset decomposition of S in what follows.

To multiply g_ie and g_je in E we write $g_ig_j = g_kn$ for some $n \in N$; then

$$(g_ie)(g_je) = (g_ig_j)e^2 = g_kne = \psi(n)(g_ke).$$

Let $\bar{S} = S/N$; then $\bar{S}$ is the set of cosets $\{g_iN\}$ given above. Define a factor set f on $\bar{S} \times \bar{S}$ via $f(g_iN, g_jN) = \psi(n)$ where $g_ig_j = g_kn$, $n \in N$. From these remarks we have identified E with $K(\bar{S})_f$ via $g_ie \leftrightarrow [g_iN]$.

Similarly let $F=eKHe$ and $T=S\cap H$. Then F may be identified with $K(\overline{T})_f$, where $\overline{T}=T/N$ and $\overline{T}$ is a subgroup of $\overline{S}$.

By the compatibility conditions (*), ζ and χ are irreducible characters of KH and KG, respectively, such that $(\psi^H, \zeta)>0$ and $(\psi^G, \chi)>0$. By Theorem A, the restrictions ζ_F of ζ to F and χ_E of χ to E are irreducible characters of F and E, respectively. Moreover, $(\psi^H, \zeta)=1$ implies that ζ_F is a linear character of F, and $(\zeta^G, \chi)\geqq 1$ implies that the restriction χ_F of χ_E to F contains ζ_F with positive multiplicity. By Theorem B (observing that $[\overline{1}]$ is the identity for E, where $\overline{1}=1\cdot N=N\in\overline{S}$) the degree $\chi_E([\overline{1}])$ of χ_E divides

$$[\overline{S}:\overline{T}]\ \mathrm{lcm}\{\mathrm{ind}_{\overline{T}}\,\sigma : \zeta^{[\sigma]} = \zeta \text{ on } K(\overline{T}^{(\sigma)})_f,\ \sigma\in\overline{S}\}, \tag{1}$$

where $\zeta^{[\sigma]}([\tau])=\zeta([\sigma][\tau][\sigma]^{-1})$ for all $\tau\in\overline{T}$.

By Theorem A we may write the block idempotents in E and F corresponding to χ_E and ζ_F, respectively, as follows:

$$\begin{aligned} e_1 &= [G:N]^{-1}\chi(1)\sum_{\sigma\in\overline{S}}\chi([\sigma]^{-1})[\sigma]\in E\\ e_2 &= [H:N]^{-1}\zeta(1)\sum_{\tau\in\overline{T}}\zeta([\tau]^{-1})[\tau]\in F.\end{aligned} \tag{2}$$

Note that N is normal in G so the terms $\mathrm{ind}_N\, g$ are all one. Also, in the identification of eg_ie with $[g_iN]$ we see that $eg_i^{-1}e$ corresponds to $[g_iN]^{-1}$.

By the development of §1, we may calculate the block idempotents e_1 and e_2 in $K(\overline{S})_f$ and $K(\overline{T})_f$ in terms of the irreducible characters χ_E and ζ_F. Indeed,

$$\begin{aligned} e_1 &= |\overline{S}|^{-1}\chi([\overline{1}])\sum_{\sigma\in\overline{S}}\chi([\sigma]^{-1})[\sigma]\\ e_2 &= |\overline{T}|^{-1}\zeta([\overline{1}])\sum_{\tau\in\overline{T}}\zeta([\tau]^{-1})[\tau].\end{aligned} \tag{3}$$

Comparing the coefficients of $[\overline{1}]$ in (2) and (3) we obtain

$$\begin{aligned} [G:N]^{-1}\chi(1)\chi([\overline{1}]) &= |\overline{S}|^{-1}\chi([\overline{1}])^2\\ [H:N]^{-1}\zeta(1)\zeta([\overline{1}]) &= |\overline{T}|^{-1}\zeta([\overline{1}])^2.\end{aligned} \tag{4}$$

But ζ_F is linear on F so $\zeta([\overline{1}])=1$. Therefore the above become

$$\begin{aligned} [G:N]^{-1}\chi(1) &= |\overline{S}|^{-1}\chi([\overline{1}])\\ [H:N]^{-1}\zeta(1) &= |\overline{T}|^{-1}.\end{aligned} \tag{5}$$

Let $l=\mathrm{lcm}\{\mathrm{ind}_{\overline{T}}\sigma : \zeta^{[\sigma]}=\zeta \text{ on } K(\overline{T}^{(\sigma)})_f,\ \sigma\in\overline{S}\}$. Then (1) above becomes $[\overline{S}:\overline{T}]l$. But $|\overline{S}|=[S:N]$, $|\overline{T}|=[T:N]$ and $N\subseteq T\subseteq S$, so that $[\bar{S}: \bar{T}] = [S: T]$. Therefore

$$\chi([\bar{1}]) \quad \text{divides} \quad [S: T]l. \tag{6}$$

By (5) this is just restating that $\chi(1)$ divides $[S: N]^{-1}[G: N][S: T]l$ which equals $[T: N]^{-1}[G: N]l$. But by the second equation of (5) this is just $\zeta(1)[G: H]l$. Therefore

$$\chi(1) \quad \text{divides} \quad \zeta(1)[G: H]l. \tag{7}$$

As in §1, one checks that

$$\begin{aligned} l &= \operatorname{lcm}\{\operatorname{ind}_{\bar{T}} \sigma: \zeta^{[\sigma]} = \zeta \text{ on } K(\bar{T}^{(\sigma)})_f,\ \sigma \in \bar{S}\} \\ &= \operatorname{lcm}\{\operatorname{ind}_T x: \zeta^x = \zeta \text{ on } T^{(x)}e,\ x \in S\}, \end{aligned}$$

where $T^{(x)}e = \{te: t \in T^{(x)}\}$. We then obtain the following result:

THEOREM C. *Let G be a finite group, N a normal subgroup of G, and let H be a subgroup of G such that $N \subseteq H \subseteq G$. Let ψ, ζ, χ, be irreducible characters of N, H, G, respectively, satisfying the compatibility conditions (*). If $S = \{g \in G: \psi^g = \psi$ on $N\}$ and $T = H \cap S$ then the degree $\chi(1)$ of χ divides*

$$[G: H]\zeta(1)\operatorname{lcm}\{\operatorname{ind}_T x: \zeta^x = \zeta \text{ on } T^{(x)}e,\ x \in S\},$$

where $T^{(x)}e = \{te: t \in T^{(x)}\}$ and $e = |N|^{-1}\sum_{n \in N} \psi(n^{-1})n$.

REMARK. Let H be a subgroup of the group G, ζ a linear character of H and χ an irreducible character of G such that $(\zeta^G, \chi) \geqq 1$. Let $N = \{1\}$ and $\psi = 1$, the identity character on N. Then $(\psi^H, \zeta) = (\psi, \zeta_N) = 1$, so the hypotheses of Theorem C are satisfied. Clearly $S = G$, $T = H$, and $e = 1$ as in Theorem C. Thus the conclusion of the theorem reads "the degree $\chi(1)$ of χ divides $[G: H]\operatorname{lcm}\{\operatorname{ind}_H x: \zeta^x = \zeta$ on $H^{(x)}$, $x \in G\}$," which is exactly the last part of Theorem A.

Note also that the theorem is a triviality if $\zeta^G = \chi$, for then $\chi(1) = [G: H]\zeta(1)$ automatically.

COROLLARY. *Hypotheses as in Theorem C. If in addition H is normal in G, then the degree $\chi(1)$ of χ divides $[G: H]\zeta(1)$.*

PROOF. Since H is normal in G, T is normal in S, and so $\operatorname{ind}_T x = 1$ for all $x \in S$. This proves the corollary.

Let G be a nonabelian group of order six, H a subgroup of order two. If χ is the character of degree two on G, then χ appears with positive multiplicity in a linear character ζ on H. But clearly in this case $\chi(1)$ does not divide $[G: H]\zeta(1)$.

It would be delightful to know if one could somehow drop the subgroup N and the linear character ψ from the hypotheses of Theorem C and yet obtain a similar result about the degree of χ.

REFERENCES

1. C. W. Curtis and T. V. Fossum, *On centralizer rings and characters of representations of finite groups*, Math. Z. **107** (1968), 402–406.

2. C. W. Curtis and I. Reiner, *Representation theory of finite groups and associative algebras*, Pure and Appl. Math., vol. **11**, Interscience, New York, 1962. MR **26** #2519.

UNIVERSITY OF UTAH