BOUNDEDNESS AND DIMENSION FOR WEIGHTED AVERAGE FUNCTIONS[1]

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ABSTRACT. The paper considers a weighted average property of the type $u(x_o) = (\int_B uwdx)/(\int_B wdx)$, B a ball in E^n with center x_o. A lemma constructing such functions is presented from which it follows that if $n=1$ and the weight function w is continuously differentiable but is not an eigenfunction of the 1-dimensional Laplace operator, then u is constant. It is also shown that if w is integrable on E^n and u is bounded above or below, u is constant.

Let D be a region in n-dimensional Euclidean space E^n. Following A. K. Bose [1] we say a *weight function* on D is a nonnegative, locally integrable function on D whose integral over any closed ball lying in D is positive.

A real valued function u has the *weighted average property* with respect to the weight function w on D if uw is locally integrable on D and, for every closed ball B lying in D with center at x_0, $u(x_0) = (\int_B uwdx)/(\int_B wdx)$. We denote by $S(w, D)$ the collection of functions satisfying the weighted average property with respect to w in D. $S(w, D)$ is a real vector space containing the constants.

Bose has shown in [1], [2] and [3]:

(i) If $n>1$ and w is an eigenfunction of the Laplace operator Δ, then the dimension, dim $S(w, D)$, of $S(w, D)$ is ∞.

(ii) If $n=2$, w is in $C^1(D)$, and w is not an eigenfunction of Δ, then $1 \leqq \dim S(w, D) \leqq 2$.

(iii) For $n>2$, there is a weight function w on E^n which is not an eigenfunction of Δ but for which dim $S(w, E^n) = \infty$.

(iv) If w is a bounded continuous weight function on E^n with a positive lower bound and u is a bounded function satisfying the weighted average property with respect to w, then u is constant.

I prove in this note:

THEOREM 1. *If D is an interval in E^1 and if w is a weight function belonging to $C^1(D)$ which is not an eigenfunction of the* 1-*dimensional*

Received by the editors March 10, 1969.

AMS Subject Classification. Primary 3111.

Key Words and Phrases. Eigenfunction of Laplace operator, dimension of space of functions satisfying a weighted average property, weight function integrable over E^n, bounded function satisfying a weighted average property.

[1] This research was supported in part by the U. S. Army Research Office in Durham, N. C.

Laplace operator, then $S(w, D)$ contains only the constants.

THEOREM 2. *If w is a weight function integrable over E^n, u is in $S(w, E^n)$, and u is bounded either above or below, then u is constant.*

The proof of Theorem 1 is based on the following;

LEMMA. *Let $w_1(x)$ be a weight function on $D_1 \subset E^\alpha$, $w_2(y)$ a weight function on $D_2 \subset E^\beta$, $u_1(x) \in S(w_1, D_1)$, and $u_2(y) \in S(w_2, D_2)$. Then $u(x, y) = u_1(x)u_2(y)$ belongs to $S(w, D_1 \times D_2)$, where $w(x, y) = w_1(x)w_2(y)$.*

PROOF. It is clear that w is a weight function and uw is locally integrable on $D_1 \times D_2$. For x in E^n, $0<r<s$, we denote by $B(x, r)$ the closed ball centered at x of radius r, and by $A(x, r, s)$ the closed annulus centered at x of radii r and s. Suppose $(x_0, y_0) \in D_1 \times D_2$, and $r>0$ such that $B((x_0, y_0), r) \subset D_1 \times D_2$. For positive integers m and p with $1 \leqq p \leqq 2^m - 1$, let

$$r_{m,p} = r(1 - p^2/2^{2m})^{1/2},$$

$$C(m, p) = B(x_0, r_{m,p}) \times A(y_0, (p-1)r/2^m, pr/2^m),$$

and

$$S_m = \cup_p C(m, p).$$

The following three statements follow from tedious but straightforward calculations:

(a) The $(\alpha+\beta)$-Lebesgue measure of $C(m, p) \cap C(m, q)$ is zero for $p \neq q$.

(b) $S(m) \subset S(m+1)$, $m \geqq 1$.

(c) Interior $B((x_0, y_0), r) \subset \cup_m S(m) \subset B((x_0, y_0), r)$.

It is easily seen that

$$u_2(y_0) \int_{A(y_0,s,t)} w_2 = \int_{A(y_0,s,t)} u_2 w_2 \quad \text{whenever} \quad B(y_0, t) \subset R_2.$$

Thus, from Fubini's Theorem,

$$u(x_0, y_0) \int_{C(m,p)} w = \int_{C(m,p)} uw \qquad \text{for all } m,\ 1 \leqq p \leqq 2^m - 1.$$

From (a) it follows that $u(x_0, y_0) \int_{S(m)} w = \int_{S(m)} uw$. Statement (b) allows us to take limits as $m \to \infty$, and using (c) also, we obtain

$$u(x_0, y_0) \int_{B((x_0,y_0),r)} w = \int_{B((x_0,y_0)\, r)} uw$$

which completes the proof.

PROOF OF THEOREM 1. Suppose $u \in S(w, D)$ and u is not constant. Let $w_2 \equiv 1$ on E^1. Then $w(x)w_2(y)$ is in $C^1(D \times E^1)$ and is not an eigenfunction of Δ. Further, $u_2(y) = y$ is in $S(w_2, E^2)$. Thus, by the lemma, each of $u(x)$, $u_2(y)$, and 1 is in $S(w(x)w_2(y), D \times E^1)$, and since these functions are linearly independent, Bose's result (ii) is contradicted.

We note that the following statement also follows easily from the lemma to Theorem 1:

If $D \subset E^n$, $n > 2$, and w is a weight function on D independent of two of the variables, then $S(w, D) = \infty$.

The author is indebted to the referee for the following proof of Theorem 2, which is shorter than the original.

PROOF OF THEOREM 2. Suppose $u \in S(w, E^n)$ and u is bounded below. Let $K > 0$ such that $v = u + K$ is positive. Then $v \in S(w, E^n)$ and, for $y \in E^n$, $R > 0$, $\int_{B(y,R)} v(x)w(x)dx = v(y)\int_{B(y,R)} w(x)dx$. Since w is integrable on E^n, vw is integrable on E^n, and, letting $R \to \infty$, $v(y) = \int v(x)w(x)dx / \int w(x)dx$, the integrals taken over all of E^n. Thus v is constant, so u is constant. If u is bounded above, consider $-u$.

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