BOUNDEDNESS AND DIMENSION FOR WEIGHTED AVERAGE FUNCTIONS

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Abstract. The paper considers a weighted average property of the type \( u(x_0) = \frac{\int_B uwdx}{\int_B wdx} \), \( B \) a ball in \( E^n \) with center \( x_0 \). A lemma constructing such functions is presented from which it follows that if \( n = 1 \) and the weight function \( w \) is continuously differentiable but is not an eigenfunction of the 1-dimensional Laplace operator, then \( u \) is constant. It is also shown that if \( w \) is integrable on \( E^n \) and \( u \) is bounded above or below, \( u \) is constant.

Let \( D \) be a region in \( n \)-dimensional Euclidean space \( E^n \). Following A. K. Bose [1] we say a weight function on \( D \) is a nonnegative, locally integrable function on \( D \) whose integral over any closed ball lying in \( D \) is positive.

A real valued function \( u \) has the weighted average property with respect to the weight function \( w \) on \( D \) if \( uw \) is locally integrable on \( D \) and, for every closed ball \( B \) lying in \( D \) with center at \( x_0 \), \( u(x_0) = \frac{\int_B uwdx}{\int_B wdx} \). We denote by \( S(w, D) \) the collection of functions satisfying the weighted average property with respect to \( w \) in \( D \). \( S(w, D) \) is a real vector space containing the constants.

Bose has shown in [1], [2] and [3]:

(i) If \( w > 1 \) and \( w \) is an eigenfunction of the Laplace operator \( \Delta \), then the dimension, \( \dim S(w, D) \), of \( S(w, D) \) is \( \infty \).

(ii) If \( n = 2 \), \( w \) is in \( C^1(D) \), and \( w \) is not an eigenfunction of \( \Delta \), then \( 1 \leq \dim S(w, D) \leq 2 \).

(iii) For \( n > 2 \), there is a weight function \( w \) on \( E^n \) which is not an eigenfunction of \( \Delta \) but for which \( \dim S(w, E^n) = \infty \).

(iv) If \( w \) is a bounded continuous weight function on \( E^n \) with a positive lower bound and \( u \) is a bounded function satisfying the weighted average property with respect to \( w \), then \( u \) is constant.

I prove in this note:

**Theorem 1.** If \( D \) is an interval in \( E^1 \) and if \( w \) is a weight function belonging to \( C^1(D) \) which is not an eigenfunction of the 1-dimensional...
Laplace operator, then \( S(w, D) \) contains only the constants.

**Theorem 2.** If \( w \) is a weight function integrable over \( E^n \), \( u \) is in \( S(w, E^n) \), and \( u \) is bounded either above or below, then \( u \) is constant.

The proof of Theorem 1 is based on the following;

**Lemma.** Let \( w_1(x) \) be a weight function on \( D_1 \subseteq E^n \), \( w_2(y) \) a weight function on \( D_2 \subseteq E^n \), \( u_1(x) \in S(w_1, D_1) \), and \( u_2(y) \in S(w_2, D_2) \). Then \( u(x, y) = u_1(x)u_2(y) \) belongs to \( S(w, D_1 \times D_2) \), where \( w(x, y) = w_1(x)w_2(y) \).

**Proof.** It is clear that \( w \) is a weight function and \( uw \) is locally integrable on \( D_1 \times D_2 \). For \( x \) in \( E^n \), \( 0 < r < s \), we denote by \( B(x, r) \) the closed ball centered at \( x \) of radius \( r \), and by \( A(x, r, s) \) the closed annulus centered at \( x \) of radii \( r \) and \( s \). Suppose \( (x_0, y_0) \in D_1 \times D_2 \), and \( r > 0 \) such that \( B((x_0, y_0), r) \subseteq D_1 \times D_2 \). For positive integers \( m \) and \( p \) with \( 1 \leq p \leq 2m - 1 \), let

\[
  r_{m,p} = r(1 - p^2/2^{2m})^{1/2},
\]

\[
  C(m, p) = B(x_0, r_{m,p}) \times A(y_0, (p - 1)r/2^m, pr/2^w),
\]

and

\[
  S_m = \bigcup_p C(m, p).
\]

The following three statements follow from tedious but straightforward calculations:

(a) The \((\alpha + \beta)\)-Lebesgue measure of \( C(m, p) \cap C(m, q) \) is zero for \( p \neq q \).

(b) \( S(m) \subseteq S(m+1), m \geq 1 \).

(c) Interior \( B((x_0, y_0), r) \subseteq U_mS(m) \subseteq B((x_0, y_0), r) \).

It is easily seen that

\[
  u_2(y_0) \int_{A(y_0, s, t)} w_2 = \int_{A(y_0, s, t)} u_2w_2 \quad \text{whenever} \quad B(y_0, t) \subseteq R_2.
\]

Thus, from Fubini’s Theorem,

\[
  u(x_0, y_0) \int_{C(m, p)} w = \int_{C(m, p)} uw \quad \text{for all} \quad m, 1 \leq p \leq 2^m - 1.
\]

From (a) it follows that \( u(x_0, y_0)\int_{S(m)} w = \int_{S(m)} uw \). Statement (b) allows us to take limits as \( m \to \infty \), and using (c) also, we obtain

\[
  u(x_0, y_0) \int_{B((x_0, y_0), r)} w = \int_{B((x_0, y_0), r)} uw
\]

which completes the proof.
Proof of Theorem 1. Suppose \( u \in S(w, D) \) and \( u \) is not constant. Let \( w_2 \equiv 1 \) on \( E^1 \). Then \( w(x)w_2(y) \) is in \( C^1(D \times E^1) \) and is not an eigenfunction of \( \Delta \). Further, \( u_2(y) = y \) is in \( S(w_2, E^2) \). Thus, by the lemma, each of \( u(x) \), \( u_2(y) \), and \( 1 \) is in \( S(w(x)w_2(y), D \times E^1) \), and since these functions are linearly independent, Bose's result (ii) is contradicted.

We note that the following statement also follows easily from the lemma to Theorem 1:

If \( D \subseteq E^n \), \( n > 2 \), and \( w \) is a weight function on \( D \) independent of two of the variables, then \( S(w, D) = \infty \).

The author is indebted to the referee for the following proof of Theorem 2, which is shorter than the original.

Proof of Theorem 2. Suppose \( u \in S(w, E^n) \) and \( u \) is bounded below. Let \( K > 0 \) such that \( v = u + K \) is positive. Then \( v \in S(w, E^n) \) and, for \( y \in E^n \), \( R > 0 \), \( \int_{B(y, R)} v(x)w(x)dx = v(y)\int_{B(y, R)} w(x)dx \). Since \( w \) is integrable on \( E^n \), \( vw \) is integrable on \( E^n \), and, letting \( R \to \infty \), \( v(y) = \int v(x)w(x)dx/\int w(x)dx \), the integrals taken over all of \( E^n \). Thus \( v \) is constant, so \( u \) is constant. If \( u \) is bounded above, consider \(-u\).

References


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