NODAL ALGEBRAS OF DIMENSION $p^3$

JERRY I. GOLDMAN

1. Introduction. The algebras discussed in this paper are members of the class $K$, of nodal noncommutative Jordan algebras, constructed as follows by Kokoris in [3]. For $p > 2$ a prime number, $F$ a field of characteristic $p$, and $n \geq 2$, let

$$B_n = F[1, x_1, \ldots, x_n]/(x_1^p, \ldots, x_n^p).$$

An algebra $A$ is in $K$ if and only if there exist $p$, $n$, and $F$ such that $A^+ = B_n$ where multiplication in $A$ is defined by

$$fg = f \cdot g + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} [x_i, x_j]$$

in terms of the (dot) product in $B_n$, and the $[x_i, x_j] = x_i x_j - x_j x_i$ are arbitrary except for the condition that at least one of these commutators be nonsingular.

Every simple nodal noncommutative Jordan algebra is in $K$, but not all the algebras in $K$ are simple [3]. Properties of $K$ have been studied in [2], [3], [4], and in several other papers, but the ideal structure of the nonsimple algebras of $K$ is unknown. In this paper we give a complete list of the ideals of a Lie-admissible algebra in $K$ with three generators.

Theorem. If $A \in K$ is Lie-admissible and of dimension $p^3$, then there exist generators $x, y, z$ such that $A = F[1] + F[x, y, z]$ (vector space direct sum) with

$$[y, x] = 1 + c \cdot x^{p-1} \cdot y^{p-1}, \quad c \in A,$$

$$[z, x] = y^{p-1} \cdot m(z), \quad m(z) \in F[1, z], \quad \text{and}$$

$$[z, y] = x^{p-1} \cdot n(z), \quad n(z) \in F[1, z].$$

If one of $m(z)$ or $n(z)$ is nonsingular, then $A$ is simple; otherwise, the proper ideals of $A$ are precisely $z \cdot A, z^2 \cdot A, \ldots, z^{p-1} \cdot A$.

2. Proof of the theorem.

Lemma 1. Let $A \in K$ have at least three generators. Then there exist generators $x$ and $y$ of $A$ satisfying

Received by the editors May 6, 1969.

1 This research was partially supported by N.S.F. grant GP 8969.
\[(2a) \quad [y, x] = 1 + c \cdot x^{p-1} \cdot y^{p-1}, \quad c \in A;\]

Further, any third generator \( z \) of \( A \) may be assumed to satisfy

\[(2b) \quad [z, x] = y^{p-1} \cdot m, \quad m \in A, \]
\[(z, y] = x^{p-1} \cdot n, \quad n \in A.\]

**Proof.** This lemma is basically due to Oehmke \([4]\); however, simplicity and Lie-admissibility of \( A \) are not needed as a careful reading of his proof (pp. 417–418) will show.

**Lemma 2.** Let \( A \in K \) be Lie-admissible and of dimension \( p^3 \). Then there exist generators \( x, y, z \) of \( A \) which satisfy \((1). If either \( m(z) \) or \( n(z) \) is nonsingular, then \( A \) is simple.

**Proof.** For any \( a \in A \), define the derivation \( D(a) \) by \( bD(a) = [b, a] \) for every \( b \in A \). Since \( A^- \) is a Lie algebra we have the Jacobi identity

\[(3) \quad zD(y)D(x) - zD(x)D(y) + yD(x)D(z) = 0.\]

Using Lemma 1 and \((2)\), substitute into \((3)\) to obtain \( x^{p-1} \cdot nD(x) - y^{p-1} \cdot mD(y) + x^{p-1} \cdot y^{p-1} \cdot cD(z) = 0 \). It is clear from \((2b)\) that \( x^{p-1} \cdot y^{p-1} \cdot cD(z) = 0 \), thus

\[(4) \quad x^{p-1} \cdot nD(x) = y^{p-1} \cdot mD(y).\]

Without loss of generality we can assume in \((2b)\) that \( m \in F[1, x, z] \) and \( n \in F[1, y, z] \). Write

\[(5) \quad m = \sum_{i,j=0}^{p-1} \alpha_{ij}x^i \cdot z^j \quad \text{and} \quad n = \sum_{i,j=0}^{p-1} \beta_{ij}y^i \cdot z^j.\]

Using \((5)\) and \((2)\), one can calculate that

\[x^{p-1} \cdot nD(x) = \sum_{j=1}^{p-1} j\beta_{0j}x^{p-1} \cdot y^{p-1} \cdot z^{j-1} \cdot m + \sum_{i=1}^{p-1} i\beta_{i0}x^{p-1} \cdot y^{i-1} \cdot z^j \]

and that

\[y^{p-1} \cdot mD(y) = \sum_{j=1}^{p-1} j\alpha_{0j}x^{p-1} \cdot y^{p-1} \cdot z^{j-1} \cdot n - \sum_{i=1}^{p-1} i\alpha_{i0}x^{i-1} \cdot y^{p-1} - \sum_{i,j=1}^{p-1} i\alpha_{ij}x^{i-1} \cdot y^{p-1} \cdot z^j.\]

Equating coefficients by \((4)\), we conclude

\[(6) \quad \alpha_{ij} = \beta_{ij} = 0, \quad 1 \leq i, j \leq p - 1,\]
\[\alpha_{i0} = \beta_{i0} = 0, \quad 1 \leq i \leq p - 1,\]
and

\[ \sum_{j=1}^{p-1} j(\alpha_{0j}m - \beta_{0j}m) \cdot z^{j-1} = 0. \]  

(7)

To eliminate double subscripts set \( \alpha_{0j} = \alpha_j \) and \( \beta_{0j} = \beta_j \) for \( 0 \leq j \leq p - 1 \). Thus, (6) implies that \( m \) and \( n \) are functions of \( z \) only:

\[ m = m(z) = \sum_{j=0}^{p-1} \alpha_j z^j \quad \text{and} \quad n = n(z) = \sum_{j=0}^{p-1} \beta_j z^j, \]

and we have (1).

To complete the proof of Lemma 2, we assume \( \alpha_0 \neq 0 \). Substitute (8) into (7) and again equate coefficients of the result to find

\[ \sum_{k=0}^{s} k(\alpha_{s-k} - \beta_{s-k}) = 0 \quad \text{for all} \quad s = 1, \ldots, p. \]

(9)

One can use the equation (9) and induction to yield the fact that \( \beta_i = (\alpha_i/\alpha_0)\beta_0, \ i = 0, \ldots, p - 1 \), and, therefore, we have

\[ \alpha_0 n(z) = \beta_0 m(z). \]

(10)

We can now proceed as in [4, p. 421] to normalize \( m(z) \) (to the form \( 1 + \beta z^{p-1} \)) and thus conclude from Theorem 1 of [2] that \( A \) is simple. This proves Lemma 2.

Remark. We can actually use (9) further to conclude \( m(z) \) and \( n(z) \) are proportional independent of whether or not one of them is nonsingular.

Lemma 3. Let \( A \in K \) be Lie-admissible and of dimension \( p^3 \). Thus, we can assume existence of generators \( x, y, \) and \( z \) satisfying (1). If \( m(z) = 0(z - A) \) and \( n(z) = 0(z - A) \), then all \( z' - A, i = 1, \ldots, p - 1 \), are proper ideals of \( A \).

Proof. We have \( [z' \cdot f, g] = z' \cdot [f, g] + if \cdot z'^{-1} \cdot [z, g] \). So, \( z' \cdot A \) is an ideal of \( A^- \). Since \( z' \cdot A \) is also an ideal of \( A^+ \), the conclusion follows.

It is apparent that Lemma 2 supplies the proof of the theorem apart from the case when \( m(z) \) and \( n(z) \) are both singular. Consequently, we now assume that \( m(z) = 0(z - A) \) and \( n(z) = 0(z - A) \) and proceed to prove that the ideals of Lemma 3 are an exhaustive list.

Completion of the proof of the theorem. Let \( I \neq 0 \) be an ideal of \( A \). By [1], \( x^{p-1} \cdot y^{p-1} \cdot z^{p-1} \in I \). Apply (1) to calculate:

\[ (x^{p-1} \cdot y^{p-1} \cdot z^{p-1}) D^{p-2}(y) = - (\phi - 1) ! x \cdot y^{p-1} \cdot z^{p-1} \]
and

$$(x \cdot y^{p-1} \cdot z^{p-1}) \cdot D^{p-2}(x) = (p - 1) \cdot x \cdot y \cdot z^{p-1}.$$  

So, \(x \cdot y \cdot z^{p-1} \in I\), which implies \(x \cdot y \cdot D(x) \cdot z^{p-1} \in I\). The nonsingularity of \(y \cdot D(x)\) implies \(x \cdot z^{p-1} \in I\). Similarly, we find \(z^{p-1} \in I\). Therefore, \(z^{p-1} \cdot A\) is the unique minimal ideal of \(A\), contained in every other nonzero ideal of \(A\).

We write \(A = A_1\) and define \(A_2\) by

$$A_2 = A_1/(z^{p-1} \cdot A_1).$$

Let \(\lambda_2\) be the natural homomorphism of \(A_1\) onto \(A_2\). The elements of \(A_2\) are of the form \(\sum \alpha(x \lambda_2)^i \cdot (y \lambda_2)^j \cdot (z \lambda_2)^k\) with \(0 \leq i, j \leq p - 1\) and \(0 \leq k \leq p - 2\) where we have again used dots to indicate product in \(A_2\) in order to simplify the notation. \(A_2\), being a homomorphic image of a nodal algebra, is a nodal algebra [5]. Since \(A_2\) inherits the basic multiplicative structure of \(A_1\), we can conclude as above that \((x \lambda_2)^{p-1} \cdot (y \lambda_2)^{p-1} \cdot (z \lambda_2)^{p-2}\) is in any ideal \(I_2\) of \(A_2\). Thus, as before, \((z \lambda_2)^{p-2} \in I_2\) and \((z \lambda_2)^{p-2} \cdot A_2\) is the unique minimal ideal of \(A_2\).

As induction hypothesis assume we have constructed a sequence of nodal algebras \(A_1, \ldots, A_{s-1}\) for \(s - 1 \leq p - 1\) and natural homomorphisms \(\lambda_3, \ldots, \lambda_{s-1}\) with \(\lambda_i\) mapping \(A_{i-1}\) onto \(A_i\). Set \(\pi_1 = \lambda_2 \lambda_3 \cdots \lambda_j\). Define \(A_s\) by

$$A_s = A_{s-1}/(z^{p-s+1} \cdot \pi_{s-1} \cdot A_{s-1}).$$

The elements of \(A_s\) are of the form \(\sum \alpha(x \pi_s)^i \cdot (y \pi_s)^j \cdot (z \pi_s)^k\) where \(0 \leq i, j \leq p - 1, 0 \leq k \leq p - s\) and \(\lambda_s\) is the natural map of \(A_{s-1}\) onto \(A_s\). Further, we can deduce as above that \((z \pi_s)^{p-s} \cdot A_s\) is the unique minimal ideal of \(A_s\) contained in all ideals of \(A_s\).

Of course, the sequence \(A_1, \ldots, A_p\) constructed above ends with \(A_p\) which is isomorphic to the nodal algebra \(F1 + F[x, y]\) where \([x, y] = 1 + \gamma x^{p-1} \cdot y^{p-1}\) and \(\gamma = c(0, 0, 0)\). Thus, \(A_p = A_{p-1}/(z \pi_{p-1} \cdot A_{p-1})\) is simple and \(z \pi_{p-1} \cdot A_{p-1}\) is the only proper ideal of \(A_{p-1}\). Therefore, there is only one proper ideal \(I_{p-2}\) of \(A_{p-2}\) which properly contains the minimal ideal \((z \pi_{p-2})\cdot A_{p-2}\) of \(A_{p-2}\) such that

$$I_{p-2}/((z \pi_{p-2}) \cdot A_{p-2}) = z \pi_{p-1} \cdot A_{p-1}.$$  

Thus, \(I_{p-2} = z \pi_{p-2} \cdot A_{p-2}\) and the only proper ideals of \(A_{p-2}\) are \(z \pi_{p-2} \cdot A_{p-2}\) and \((z \pi_{p-2})^2 \cdot A_{p-2}\).

Now suppose by induction that the only proper ideals of \(A_2\) are \(z \pi_2 \cdot A_2, (z \pi_2)^2 \cdot A_2, \ldots, (z \pi_2)^{p-2} \cdot A_2\). Therefore, the only proper ideals of \(A\), which properly contain \(z^{p-1} \cdot A\), must be \(I_1^{(1)}, I_1^{(2)}, \ldots, I_1^{(p-2)}\) where
\[ I_1^{(j)} = z^j \cdot A_1 = z \cdot z^{j-1} \cdot A_1. \]

Clearly, \( I_1^{(j)} = z^j \cdot A_1 \) for \( j = 1, \ldots, p-2 \). Therefore, the only proper ideals of \( A_1 = A \) are \( z \cdot A, z^2 \cdot A, \ldots, z^{p-1} \cdot A \) and the theorem is proved.

References


DePaul University