A TWO PARAMETER PERTURBATION ESTIMATE[1]

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1. Consider the boundary value problem

$$\epsilon Au + C(\mu u) + Bu = f \equiv f_{\epsilon,\mu}$$

for $0<\epsilon\leqq\epsilon_0$ and $0<\mu\leqq\mu_0$, where A and B are linear elliptic differential operators of respective orders $m'>m$ over a bounded domain D; and C is a quasilinear differential operator of the form

$$C(v) = \sum_{|i|\leqq m} (-1)^{|i|} D_i c_i(x, v, Dv, \cdots, D^m v), \qquad x \in D,$$

with $C(0)=0$. The solution u of the above equation is to be compared with the solution u_0 of $Bu_0=f_0$ where $f\to f_0$ as $\epsilon\downarrow 0$ and $\mu\downarrow 0$. In particular, bounds of the form $|u-u_0|_{m,D}=o(\epsilon^\tau)+O(\mu)$ for the norm of $u-u_0$ in the Bessel potential space $P^m(D)$ will be derived, assuming a like bound for the $L^2(D)$ norm of $f-f_0$.

For the linear problem obtained by setting $C=0$ above, corresponding bounds have been obtained by Friedman [**7**], Greenlee [**9**], and Huet [**13**], [**14**]. These results supplement earlier work of Višik and Lyusternik [**23**], Huet [**11**], [**12**], and Ton [**21**]. Singular perturbation problems for nonlinear elliptic and parabolic equations have been considered by Ton [**22**]. In [**17**], [**18**], [**19**], O'Malley has studied multiparameter singular perturbation problems. An extensive bibliography of the literature on singular perturbation is contained in O'Malley [**20**].

In this paper a perturbation theorem for a functional equation in abstract Hilbert space is proven. The theorem is then applied to differential problems of the type described above. The methods are similar to those used in [**9**].

2. Let V and V_0 be complex Hilbert spaces with $V \subset_c V_0$, i.e., V is a vector subspace of V_0 and the injection of V into V_0 is continuous, and V dense in V_0. Denote by $|v|_V$, $(v, w)_V$, $|v|_0$, $(v, w)_0$ the norms and inner products in V and V_0 respectively. Let $a(v, w)$ be a continuous Hermitian bilinear (sesquilinear) form on V and let $b(v, w)$ be a continuous Hermitian bilinear form on V_0. Further assume that:

(1) there exists $\beta>0$ such that

$$|b(v, v)| \geqq \beta |v|_0^2 \qquad \text{for all } v \in V_0.$$

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Now let $v, w \to c(v, w): V_0 \times V_0 \to \mathbf{C}$ (the complex field) satisfy:

(2) for each fixed $v \in V_0$, $w \to c(v, w)$ is antilinear;

(3) $c(0, v) = 0$ for all $v \in V_0$;

(4) for each closed ball $S \subset V_0$ there exists $K_S > 0$ such that

$$|c(v, z) - c(w, z)| \leqq K_S |v - w|_0 |z|_0 \qquad \text{for all } v, w \in S, z \in V_0;$$

and

(5) for $0 < \epsilon \leqq \epsilon_0$ and $0 < \mu \leqq \mu_0$ there exist $\alpha(\epsilon) > 0$, $\alpha(\epsilon) \to 0$ as $\epsilon \downarrow 0$, and $\delta > 0$ such that

$$|\epsilon a(v - w, v - w) + c(\mu v, v - w) - c(\mu w, v - w) + b(v - w, v - w)| \geqq \alpha(\epsilon) |v - w|_V^2 + \delta |v - w|_0^2 \qquad \text{for all } v, w \in V.$$

Observe that (3) and (5) imply:

(6) for $0 < \epsilon \leqq \epsilon_0$,

$$|\epsilon a(v, v) + b(v, v)| \geqq \alpha(\epsilon) |v|_V^2 + \delta |v|_0^2 \qquad \text{for all } v \in V.$$

Let V_0^* be the antidual of V_0, i.e. the Hilbert space of continuous antilinear functionals on V_0, with the usual norm, $\|L\| = \sup\{|L(v)| : v \in V_0 \text{ and } |v|_0 \leqq 1\}$. Let $L_{\epsilon,\mu} \equiv L$, $0 < \epsilon \leqq \epsilon_0$, $0 < \mu \leqq \mu_0$, and L_0 be given in V_0^*. It follows from the Lax-Milgram lemma that the equation

$$b(u_0, v) = L_0(v) \qquad \text{for all } v \in V_0 \tag{7}$$

has a unique solution $u_0 \in V_0$. Furthermore, it follows from a theorem of Zarantonello [**24**], [**25**] (cf. also Browder [**5**], [**6**]) that the equations

$$\epsilon a(x, v) + c(\mu x, v) + b(x, v) = L_0(v) \qquad \text{for all } v \in V, \tag{8}$$

and

$$\epsilon a(u, v) + c(\mu u, v) + b(u, v) = L(v) \qquad \text{for all } v \in V, \tag{9}$$

have unique solutions x and u respectively, in V. In each of (8), (9) it is assumed that $0 < \epsilon \leqq \epsilon_0$ and $0 < \mu \leqq \mu_0$.

Denote by $\mathfrak{A}$ the linear operator defined by

$$b(\mathfrak{A} v, w) = a(v, w) \qquad \text{for all } w \in V$$

on

$$D(\mathfrak{A}) = \{v \in V : w \to a(v, w) \text{ is continuous on } V \text{ in the topology of } V_0\}.$$

$\mathfrak{A}$ is a closed densely defined operator in V_0 (cf. [**9**, Proposition 2.1]). Moreover, $D(\mathfrak{A})$, provided with the graph norm $(|v|_0^2 + |\mathfrak{A} v|_0^2)^{1/2}$, is a

Hilbert space, V_1, which is dense in V and whose norm and inner product will be denoted by $|v|_1$, $(v, w)_1$, respectively. The interpolation spaces by quadratic interpolation between V_1 and V_0 will be denoted by V_τ, $0 \leqq \tau \leqq 1$ (cf. Lions [**15**] or Adams, Aronszajn, and Hanna [**1**, Appendix 1]).

For a real valued function $g(\epsilon, \mu)$ the notation $g(\epsilon, \mu) = o(\epsilon) + O(\mu)$ will signify that $|g|$ is dominated by the sum of a function of ϵ which is $o(\epsilon)$ and a function of μ which is $O(\mu)$. Then the following rate of convergence theorem describes the behavior of u (and x) as $\epsilon \downarrow 0$ and $\mu \downarrow 0$.

THEOREM. *Assume hypotheses* (1)–(5) *and let* u_0, u *be the solutions of* (7), (9) *respectively. Then one has*:

(i) *if* $u_0 \in V_1 = D(\mathfrak{A})$ *and* $\|L - L_0\| = O(\epsilon) + O(\mu)$ *as* $\epsilon \downarrow 0$, $\mu \downarrow 0$, *then*

$$|u - u_0|_0 = O(\epsilon) + O(\mu) \qquad \textit{as } \epsilon \downarrow 0, \quad \mu \downarrow 0;$$

(ii) *if for fixed* $\tau \in [0, 1)$, $u \in V_\tau$ *and* $\|L - L_0\| = o(\epsilon^\tau) + O(\mu)$ *as* $\epsilon \downarrow 0$, $\mu \downarrow 0$, *then*

$$|u - u_0|_0 = o(\epsilon^\tau) + O(\mu) \qquad \textit{as } \epsilon \downarrow 0, \quad \mu \downarrow 0.$$

PROOF. Subtraction of (8) from (9) yields

$$\epsilon a(u - x, v) + c(\mu u, v) - c(\mu x, v) + b(u - x, v) = (L - L_0)(v)$$

for all $v \in V$. By letting $v = u - x$ it follows from (5) that

$$\alpha(\epsilon) |u - x|_V^2 + \delta |u - x|_0^2 \leqq \|L - L_0\| \cdot |u - x|_0.$$

Hence $|u - x|_0 \leqq (1/\delta)\|L - L_0\|$, and so

$$(10) \qquad |u - u_0|_0 \leqq (1/\delta)\|L - L_0\| + |x - u_0|_0.$$

It is thus sufficient to prove that (i) *and* (ii) *hold with* u *replaced by* x.

For this purpose let $S = (\mathfrak{A}^* \mathfrak{A})^{1/2}$ where $\mathfrak{A}^*$ is the adjoint of $\mathfrak{A}$ in V_0 and let Γ be the continuous (nonlinear) operator defined on V_0 by $b(\Gamma v, w) = c(v, w)$ for all $w \in V_0$. It follows from Zarantonello's theorem [**24**], [**25**], [**5**] that $(\epsilon\mathfrak{A} + \Gamma\mu + I)^{-1}$ is a continuous operator on V_0 and, by (8) and the definitions of $\mathfrak{A}$ and Γ, that x is the unique solution in $V_1 = D(\mathfrak{A})$ of

$$(11) \qquad (\epsilon\mathfrak{A} + \Gamma\mu + I)x = u_0.$$

Now $(\epsilon S + \mu + I)^{-1}$ is a bounded linear operator on V_0. Let y be the unique solution in $D(S) = D(\mathfrak{A})$ of

$$(12) \qquad (\epsilon S + \mu + I)y = u_0.$$

An estimate of the form

$$|x-u_0|_0 \leqq (\text{constant})\,|y-u_0|_0, \qquad \epsilon \in (0, \epsilon_0], \quad \mu \in (0, \mu_0],$$

will now be derived.

First observe that by (6), if M is a bound for $b(v, w)$ then

$$|(\epsilon\mathfrak{A}+I)^{-1}v|_0 \leqq (M/\delta)\,|v|_0, \qquad v \in V_0. \tag{13}$$

Also,

$$|(\epsilon S+\mu)^{-1}v|_0 \leqq (1/\mu)\,|v|_0, \qquad v \in V_0. \tag{14}$$

Now (11) implies that

$$\begin{aligned} x &= (\epsilon\mathfrak{A}+I)^{-1}u_0 - (\epsilon\mathfrak{A}+I)^{-1}\Gamma\mu x \\ &= u_0 - \epsilon\mathfrak{A}(\epsilon\mathfrak{A}+I)^{-1}u_0 - (\epsilon\mathfrak{A}+I)^{-1}\Gamma\mu x, \end{aligned}$$

and so,

$$x-u_0 = -\,\epsilon\mathfrak{A}(\epsilon\mathfrak{A}+I)^{-1}u_0 - (\epsilon\mathfrak{A}+I)^{-1}\Gamma\mu(\epsilon\mathfrak{A}+\Gamma\mu+I)^{-1}u_0. \tag{15}$$

Furthermore (12) yields

$$y = u_0 - (\epsilon S+\mu)(\epsilon S+\mu+I)^{-1}u_0 \tag{16}$$

and so

$$u_0 = -\,(\epsilon S+\mu+I)(\epsilon S+\mu)^{-1}(y-u_0). \tag{17}$$

Hence, by (15) and (17),

$$\begin{aligned} x-u_0 \\ &= \epsilon\mathfrak{A}(\epsilon\mathfrak{A}+I)^{-1}(\epsilon S+\mu+I)(\epsilon S+\mu)^{-1}(y-u_0) \\ &\quad - (\epsilon\mathfrak{A}+I)^{-1}\Gamma\mu(\epsilon\mathfrak{A}+\Gamma\mu+I)^{-1}(\epsilon S+\mu+I)(\epsilon S+\mu)^{-1}(u_0-y). \end{aligned}$$

Thus, using (13),

$$\begin{aligned} |x-u_0|_0 \\ &\leqq |\epsilon\mathfrak{A}(\epsilon\mathfrak{A}+I)^{-1}(y-u_0)|_0 + |\epsilon\mathfrak{A}(\epsilon\mathfrak{A}+I)^{-1}(\epsilon S+\mu)^{-1}(y-u_0)|_0 \\ &\quad + |(\epsilon\mathfrak{A}+I)^{-1}\Gamma\mu(\epsilon\mathfrak{A}+\Gamma\mu+I)^{-1}[I+(\epsilon S+\mu)^{-1}](u_0-y)|_0 \\ &= |y-u_0-(\epsilon\mathfrak{A}+I)^{-1}(y-u_0)|_0 \\ &\quad + |(\epsilon\mathfrak{A}+I)^{-1}\epsilon\mathfrak{A}(\epsilon S+\mu)^{-1}(y-u_0)|_0 \\ &\quad + |(\epsilon\mathfrak{A}+I)^{-1}\Gamma\mu(\epsilon\mathfrak{A}+\Gamma\mu+I)^{-1}[I+(\epsilon S+\mu)^{-1}](u_0-y)|_0 \\ &\leqq |y-u_0|_0 + (M/\delta)\,|y-u_0|_0 + (M/\delta)\,|\epsilon\mathfrak{A}(\epsilon S+\mu)^{-1}(y-u_0)|_0 \\ &\quad + (M/\delta)\,|\Gamma\mu(\epsilon\mathfrak{A}+\Gamma\mu+I)^{-1}[I+(\epsilon S+\mu)^{-1}](u_0-y)|_0 \\ &\leqq [1+(2M/\delta)]\,|y-u_0|_0 \\ &\quad + (M/\delta)\,|\Gamma\mu(\epsilon\mathfrak{A}+\Gamma\mu+I)^{-1}[I+(\epsilon S+\mu)^{-1}](u_0-y)|_0. \end{aligned}$$

It follows from (16) that $|u_0-y|_0 \leqq |u_0|_0$ and so (5) and (14) imply that

$$|\mu(\epsilon\mathcal{A} + \Gamma\mu + I)^{-1}[I + (\epsilon S + \mu)^{-1}](u_0 - y)|_0 \leqq (M/\delta)[\mu + 1]|u_0|_0.$$

Thus letting $S = \{v \in V_0 : |v|_0 \leqq (M/\delta)[\mu_0+1]|u_0|_0\}$ it follows from (1), (3), (4), and the above that with $K = K_S$,

$$|x - u_0|_0 \leqq [1 + (2M/\delta) + (KM^2/\beta\delta^2)(\mu_0 + 1)]|y - u_0|_0, \tag{18}$$
$$\epsilon \in (0, \epsilon_0], \qquad \mu \in (0, \mu_0].$$

It remains to estimate $|y-u_0|_0$. Let $u \in V_\tau$ with $\tau \in [0, 1]$ and let E be the resolution of the identity for the selfadjoint operator S. Then

$$\begin{aligned} |y - u_0|_0^2 &= |[(\epsilon S + \mu + I)^{-1} - I]u_0|_0^2 \\ &= \int_0^\infty \frac{(\epsilon\lambda + \mu)^2}{(\epsilon\lambda + \mu + 1)^2}(E(d\lambda)u_0, u_0)_0 \\ &\leqq 2\epsilon^{2\tau}\int_0^\infty \lambda^{2\tau}\frac{(\epsilon\lambda)^{2-2\tau}}{(\epsilon\lambda + 1)^{2-2\tau}}(E(d\lambda)u_0, u_0)_0 + 2\mu^2|u_0|_0^2. \end{aligned}$$

But $u \in V_\tau = D(S^\tau)$ if and only if $\int_0^\infty \lambda^{2\tau}(E(d\lambda)u_0, u_0)_0 < \infty$ and so Lebesgue's dominated convergence theorem yields

$$\begin{aligned} |y - u_0|_0 &= o(\epsilon^\tau) + O(\mu), \qquad \tau \in [0, 1), \quad \epsilon \downarrow 0, \quad \mu \downarrow 0, \\ &= O(\epsilon) + O(\mu), \qquad \tau = 1, \quad \epsilon \downarrow 0, \quad \mu \downarrow 0. \end{aligned} \tag{19}$$

The theorem follows from (10), (18), and (19).

Observe that an explicit estimate for $|u-u_0|_0$ in terms of given parameters, $|u|_0$, and $|u|_\tau$ is obtainable from the proof of the theorem.

3. The above theorem will now be applied in the context of perturbation problems for elliptic partial differential equations. The terminology of the theory of Bessel potentials will be used (cf. Aronszajn and Smith [3], and Adams, Aronszajn, and Smith [2]).

Let $m' > m$ be positive integers and let $D \subset R^n$ be a bounded domain of class $C^{2m'}$. Recall that for such domains and any $\alpha > 0$ the Bessel potential spaces $P^\alpha(D)$ and $\check{P}^\alpha(D)$ coincide up to equivalent norms (cf. [2]). Let V be a closed subspace of $P^{m'}(D)$ containing $C_0^\infty(D)$. Further let V_0 be a closed subspace of $P^m(D)$ such that $V \subset V_0$ and V is dense in V_0. For $v, w \in V$, let

$$a(v, w) = \sum_{|i|,|j| \leq m'} \int_D a_{ij}(x) D_j v \overline{D_i w} dx$$

where $a_{ij} \in C^{|i|}(\overline{D})$. For $v, w \in V_0$, let

$$b(v, w) = \sum_{|i|,|j| \leq m} \int_D b_{ij}(x) D_j v \overline{D_i w} dx$$

with $b_{ij} \in C^{|i|}(\overline{D})$, and let

$$c(v, w) = \sum_{|i| \leq m} \int_D c_i(x, \xi(v)(x)) \overline{D_i w} dx$$

where $c_i(x, \xi)$ is continuous in ξ for fixed $x \in D$ and measurable in x for fixed ξ, and $\xi = (\xi_i)_{|i| \leq m}$, $\xi_i(v) = D_i v$. Assume moreover that $c_i(x, 0) \equiv 0$ and that

$$| c_i(x, \zeta_1, \eta_1) - c_i(x, \zeta_2, \eta_2) | \leqq K_1(\zeta_1, \zeta_2) | \zeta_1 - \zeta_2 | + K_2 | \eta_1 - \eta_2 |$$

where K_1 is positive and continuous, K_2 is a positive constant, and, in the decomposition $\xi = (\zeta, \eta)$, the ζ variables correspond to derivatives of order $\leqq k < m - (n/2)$. Then (2) and (3) are satisfied and it follows from the Sobolev theorems (cf. [**3**, p. 424]) that (4) is satisfied. Assume that (1) and (5) also hold.

Let $f_{\epsilon,\mu} \equiv f$, $f_0 \in L^2(D)$ and let $|v|_{0,D}$, $(v, w)_{0,D}$, $|v|_{m,D}$, $(v, w)_{m,D}$ denote the norms and inner products in $L^2(D)$ and $P^m(D)$ respectively. It now follows that there is a unique solution $u \in V$ of

$$\epsilon a(u, v) + c(\mu u, v) + b(u, v) = (f, v)_{0,D} \equiv L(v), \qquad v \in V,$$

and a unique solution $u_0 \in V_0$ of

$$b(u_0, v) = (f_0, v)_{0,D} \equiv L_0(v), \qquad v \in V_0.$$

Dirichlet problems. Let $V = P_0^{m'}(D) =$ closure of $C_0^\infty(D)$ in $P^{m'}(D)$, and $V_0 = P_0^m(D)$. Then according to Nirenberg [**16**], $u_0 \in P^{2m}(D) \cap P_0^m(D)$, and according to [**9**, Proposition 6.1 and Theorem 6.2], $V_1 = D(\mathfrak{A})$ is $P^{2m'-m}(D) \cap P_0^{m'}(D)$ with an equivalent norm. It follows from the interpolation results of [**9**] or Grisvard [**10**] (cf. also Fujiwara [**8**]) that for $0 \leqq \tau < 1/4(m'-m)$, V_τ is $P^{m+2(m'-m)\tau}(D) \cap P_0^m(D)$ with an equivalent norm. Thus since $|f - f_0|_{0,D}$ dominates $\|L - L_0\|$ (cf. [**9**, Theorem 4.1]), it follows from the above theorem that if for all $\tau < 1/4(m'-m)$, $|f - f_0|_{0,D} = o(\epsilon^\tau) + O(\mu)$ as $\epsilon \downarrow 0$, $\mu \downarrow 0$, then

$$| u - u_0 |_{m,D} = o(\epsilon^\tau) + O(\mu) \qquad \text{as } \epsilon \downarrow 0, \quad \mu \downarrow 0$$

for all $\tau < 1/4(m'-m)$.

Observe that the statement $u \in D(\mathfrak{A}) = P^{2m'-m}(D) \cap P_0^m(D)$ constitutes a theorem on the regularity of weak solutions of the semilinear partial differential equations under consideration. An earlier regularity theorem for such equations is contained in Browder [4]. A similar comment applies to the next example.

NEUMANN PROBLEMS. Let $V=P^{m'}(D)$ and $V_0=P^m(D)$. Then according to [16], $u_0\in P^{2m}(D)$ and by the same line of reasoning as used in the previous example, $V_1=D(\mathfrak{A})$ is a closed subspace of $P^{2m'-m}(D)$ determined by natural boundary conditions of orders $m', \cdots, 2m'-m-1$. It follows from the interpolation results of [10] (cf. also [8]) that for $0\leqq\tau<1/2+1/4(m'-m)$, V_τ is $P^{m+2(m'-m)\tau}$ with an equivalent norm. Thus if $2m>m'$ and $|f-f_0|_{0,D}=o(\epsilon^\tau)+O(\mu)$ as $\epsilon\downarrow 0$, $\mu\downarrow 0$ for all $\tau<1/2+1/4(m'-m)$, then

$$|u-u_0|_{m,D}=o(\epsilon^\tau)+O(\mu) \qquad \text{as } \epsilon\downarrow 0, \quad \mu\downarrow 0$$

for all $\tau<1/2+1/4(m'-m)$. The same conclusion holds in the case $2m\leqq m'$ if the functions $b_{ij}\in C^{|i|+m'-2m+1}(\overline{D})$ so that $u_0\in P^{m'+1}(D)$.

Other differential problems with smooth boundary conditions which fit within the above framework can be treated similarly.

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