NONNEGATIVE EXPANSIONS OF POLYNOMIALS

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ABSTRACT. Consider expanding a set of orthogonal polynomials $\{q_i(x)\}$ as a combination of orthogonal polynomials $\{p_i(x)\}$. A sufficient condition to ensure that the expansion coefficients are nonnegative is derived. It is shown by counter-example that it is not a necessary condition.

Often, when working with a nonclassical set of orthogonal polynomials $\{q_i(x)\}$, with orthogonality relation $[q_i, q_j]=0$, $i\neq j$ for an inner product $[\ ,\ ]$, it is desirable to consider each $q_i(x)$ as an expansion in terms of known polynomials $\{p_i(x)\}$, with orthogonality relation $(p_i, p_j)=0$, $i\neq j$.

Frequently, knowing that the expansion coefficients are nonnegative allows us to infer properties of $q_i(x)$ polynomials from the $p_i(x)$ set.

For example, if the $q_i(x)$, on $[-1, 1]$, have a nonnegative expansion in $T_i(x)$ (the first kind Chebyshev polynomials), we have a nonnegative cosine expansion for $q_i(\cos\theta)$ and can infer that $q_i(x)$ assumes its maximum on $[-1, 1]$, at $+1$.

We will assume that $[f, g]=[g, f]$, a real inner product, and that the $q_i(x)$ and $p_i(x)$ polynomials have positive leading coefficients. Then we show

THEOREM. *If* $[p_i, p_j]\leqq 0$, $i\neq j$, $i, j=0, 1, \cdots, n$, *then in the expansion*

$$q_n(x) = \sum_{i=0}^{n} \alpha_i^{(n)} p_i(x) \tag{1}$$

the coefficients $\alpha_i^{(n)}\geqq 0$, $i=0, 1, \cdots, n$.

PROOF. Note first that $\alpha_n^{(n)}>0$, since p_n and q_n have positive leading coefficients. We let $[p_i, p_j]=I_{ij}$ and note that $I_{ij}=I_{ji}$.

By the orthogonality, for $k<n$,

$$0 = [q_n, p_k] = \sum_{i=0}^{n} \alpha_i^{(n)} [p_i, p_k]$$

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so that

$$\sum_{i=0}^{n} I_{ki}\alpha_i^{(n)} = 0, \qquad k = 0, 1, \cdots, n-1.$$

Further

$$[q_n, p_n] = \sum_{i=0}^{n} I_{ni}\alpha_i^{(n)} = \beta > 0.$$

This system will have a nonnegative solution if and only if the system

$$\begin{pmatrix} I_{00}, I_{01}, \cdots, I_{0n} \\ \\ \\ \\ I_{n0}, I_{n1}, \cdots, I_{nn} \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \cdot \\ \cdot \\ \alpha_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \\ 1 \end{pmatrix}$$

has a nonnegative solution. But the matrix is a Gram matrix, $([p_i, p_j])$, (see Davis [**1**, p. 176]) and is symmetric, positive definite. Further, all off-diagonal elements are nonpositive, so it is a Stieltjes matrix (see Varga [**3**, p. 85], and by corollary 3, of Varga [**3**, p. 85]) has a nonnegative inverse. Thus, the system has a nonnegative solution, so that the expansion (1) will have nonnegative coefficients. Q.E.D.

We show that the condition is not necessary by considering the Jacobi polynomials $P_n^{(1/2,-1/2)}(x)$. We will assume the normalization (see Szegö [**2**, p. 60])

$$\begin{aligned} W_n(x) &= \sin\{(2n+1/2)\theta\}/\sin(\theta/2), \qquad \theta = \cos^{-1} x \\ &= 2\sin(n+1/2)\theta\cos(\theta/2)/\sin\theta \\ &= U_n(x) + U_{n-1}(x) \end{aligned}$$

where $U_n(x)$ are the Chebyshev polynomials of the second kind. Here, we have a nonnegative expansion of $W_n(x)$ in terms of $U_n(x)$. We compute $[U_0, U_2]$, where for [,] we take the Jacobi inner product

$$[f, g] = \int_{-1}^{1} f(x)g(x)\left(\frac{1-x}{1+x}\right)^{1/2} dx,$$

$$[U_0, U_2] = \int_{-1}^{1} U_2(x)\left(\frac{1-x}{1+x}\right)^{1/2} dx.$$

Putting $x=\cos\theta$,

$$[U_0, U_2] = \int_0^\pi \frac{\sin 3\theta(1-\cos\theta)^{1/2}}{(1+\cos\theta)^{1/2}}\,d\theta = \int_0^\pi \frac{(\sin 3\theta)(1-\cos\theta)d\theta}{\sin\theta} = \pi.$$

Thus, the off-diagonal element of the Gram matrix, $[U_0, U_2]$ is positive, yet, for $W_2(x)$, we have a nonnegative expansion in terms of $U_0(x)$, $U_1(x)$, $U_2(x)$.

Finally, we show the condition to be applicable, giving an example involving classical polynomials. (The condition has also been used in quite nonclassical cases, where it appeared to be the only way of obtaining nonnegative expansions.) Since

$$\begin{aligned} \int_{-1}^1 T_m(x)dx &= -2/(m^2-1), && m \text{ even} \\ &= 0, && n \text{ odd} \end{aligned}$$

and since

$$T_m(x)T_n(x) = \tfrac{1}{2}[T_{n+m}(x) + T_{n-m}(x)], \qquad n \geqq m$$

we see immediately that, for $i \neq j$,

$$\int_{-1}^1 T_i(x)T_j(x)dx \leqq 0, \qquad i, j = 0, 1, \cdots, n,$$

so that, as is well known, the Legendre polynomials have a nonnegative expansion in Chebyshev polynomials.

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