BOUNDED POINT DERIVATIONS AND REPRESENTING MEASURES ON $R(X)$

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Let A be a function algebra on X and let p be a point of X. A point derivation D on A at p is a linear functional (not necessarily continuous) on A satisfying

$$D(fg) = D(f)g(p) + f(p)D(g), \qquad \text{for all } f, g \in A.$$

If D is continuous, D is called a bounded point derivation.

For a compact set X in the complex plane C we use $R_0(X)$ to denote the algebra of all rational functions having no poles on X. Denote by $R(X)$ the function algebra consisting of all functions which are uniform limits on X of functions in $R_0(X)$.

If p is a point of X it is easy to see that there is a bounded point derivation at p on $R(X)$ if and only if, for all f in $R_0(X)$,

$$|f'(p)| \leqq M\|f\| \qquad \text{for some constant } M$$
$$(\|f\| = \sup\{|f(z)| : z \in X\}).$$

Recently Hallstrom [2], using the techniques of Melnikov [3], has given a very nice characterization, expressed in terms of analytic capacity, describing when points in $R(X)$ admit bounded point derivations. He also discussed the notion of point derivations of higher orders on $R(X)$. For a positive integer k, a bounded point derivation of order k on $R(X)$ at p is said to exist if there is a constant M such that, for all f in $R_0(X)$,

$$|f^{(k)}(p)| \leqq M\|f\|,$$

where $f^{(k)}(p)$ denotes the kth derivative of f at p. Hallstrom extended his characterization for bounded point derivations of order 1 to include those of order k.

In this note we obtain a similar characterization in terms of representing measures. That is, we show there exist bounded point derivations on $R(X)$ of order k at a point p if and only if p has (complex) representing measures of a certain type. By a measure on X we mean a finite complex Baire measure on X. If μ is a measure, $|\mu|$ denotes its total variation. We call μ a (complex) representing measure for a point p in X if

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$$f(p) = \int f d\mu \qquad \text{for every } f \text{ in } R(X).$$

THEOREM. *There exists a nonzero bounded point derivation on $R(X)$ of order $k \geqq 1$ at a point p in X if and only if there is a (complex) representing measure μ_p for p satisfying*

$$\int d|\mu_p|(z)/|z-p|^k < \infty.$$

PROOF. Suppose there exists a nonzero bounded point derivation on $R(X)$ of order k at p. Then, for all f in $R_0(X)$,

$$|f^{(k)}(p)| \leqq M\|f\| \qquad \text{for some constant } M.$$

Hence there exists a measure μ on X with

$$f^{(k)}(p) = \int f(z)d\mu(z), \qquad f \in R_0(X).$$

Let $g(z) = (z-p)^k \in R_0(X)$. Then for each $f \in R_0(X)$,

$$(fg)^{(k)}(p) = k!f(p).$$

Thus

$$f(p) = \frac{1}{k!}\int f(z)(z-p)^k d\mu(z)$$

for all f in $R_0(X)$. Letting $\mu_p = (1/k!)(z-p)^k\mu$ and taking uniform limits, we obtain

$$f(p) = \int f(z)d\mu_p(z)$$

for all f in $R(X)$. Clearly

$$\int \frac{d|\mu_p|(z)}{|z-p|^k} < \infty,$$

and we have the desired representing measure.

Conversely, suppose such a measure μ_p exists. Since μ_p is a representing measure it is an easy computation to show that no linear combination of the measures μ_p, $(z-p)^{-1}\mu_p, \cdots, (z-p)^{-k}\mu_p$ is orthogonal to $R(X)$—integration against an appropriate power of $z-p$ is nonzero. However, every $f \in R_0(X)$ satisfying $f(p) = f'(p) = \cdots = f^{(k)}(p) = 0$ is annihilated by these measures. It follows that the closure of such functions has codimension $k+1$ in $R(X)$. Thus for each j, $1 \leqq j \leqq k$, the functional $f \to f^{(j)}(p)$ must extend continuously to $R(X)$, as required.

In [4] Wermer constructed a compact set X in C for which $R(X)$ is proper in $C(X)$, but $R(X)$ admits no bounded point derivations. Using Wermer's example we find the following two corollaries surprising.

COROLLARY. *There exists a compact set X in C with $R(X) \neq C(X)$ but no point of X has a representing measure μ_p satisfying*

$$\int \frac{d\mu_p(z)}{|z - p|} < \infty.$$

COROLLARY. *There exists a compact set X in C with $R(X) \neq C(X)$ and with the following property: If P is a nontrivial part of X and $\{\mu_q\}$ is any family of representing measures in one-to-one correspondence with the points of P, then for each $p \in P$, $\{q \in P : \mu_q$ is boundedly absolutely continuous with respect to $\mu_p\}$ has zero planar Lebesgue measure.*

PROOF. Use Wermer's example together with the well-known fact that, for any measure μ with compact support,

$$\int \frac{d|\mu|(z)}{|z - t|} < \infty$$

almost everywhere in t with respect to planar Lebesgue measure.

This example should be contrasted with the case where X has non empty interior. For then the points in each component of int X always have representing measures which are pairwise mutually boundedly absolutely continuous (see [1]).

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