IRREGULAR INVARIANT MEASURES RELATED TO HAAR MEASURE

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Abstract. Let \( G \) be a locally compact nondiscrete group, and let \( n \) be a Haar measure on an open subgroup of \( G \). It is not hard to show that \( n \) must be the restriction of a Haar measure \( v \) on all of \( G \). Here we show that there exists a translation invariant measure \( \mu \) (found by extending \( n \) to the cosets of \( H \) in a natural way) which agrees with \( v \) on, for example, \( \sigma \)-finite sets, open sets, and subsets of \( H \). Although \( v \) can be computed from \( \mu \) in a relatively simple manner, the two measures are not equal in general. In fact, there is an extreme case, namely when \( H \) is not \( \sigma \)-compact and has uncountably many cosets, in which \( \mu \) fails very badly to be regular—there are closed sets on which \( \mu \) is not inner regular and (other) closed sets on which \( \mu \) is not outer regular. One condition sufficient for this extreme case to be possible is when \( G \) is Abelian and not \( \sigma \)-compact.

1. Definitions and notation. Let \( \mu \) be a (nonnegative, countably additive) measure defined on a \( \sigma \)-algebra \( M \) of subsets of a topological space \( X \). If \( S \) is in \( M \), we say that \( \mu \) is inner regular on \( S \) if

\[
\mu S = \sup \{ \mu C : C \subseteq M, C \text{ compact}, C \subseteq S \}.
\]

We say that \( \mu \) is outer regular on \( S \) if

\[
\mu S = \inf \{ \mu U : U \subseteq M, U \text{ open}, U \supseteq S \}.
\]

Following [2, 11.34], we say that \( \mu \) is regular if it is outer regular on every set in \( M \) and inner regular on every open set in \( M \), and if every compact set in \( M \) has finite measure.

By Haar measure on a locally compact group \( G \), we mean a left Haar measure as defined in, e.g., [2]; that is to say, a left-translation invariant, regular, nondegenerate measure on a \( \sigma \)-algebra \( M(G) \) of subsets of \( G \). \( M(G) \) contains all the closed subsets of \( G \) and consists of all the sets which are measurable with respect to the Carathéodory outer measure associated with the measure.

If \( H \) is a subgroup of \( G \) (not necessarily normal), \( G/H \) denotes the space of left cosets of \( H \) in \( G \). If \( S \) is a set, \( P(S) \) denotes the collection of all subsets of \( S \); \( |S| \) denotes the cardinality of \( S \).

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2. **Lemma 1.** Let \( X \) be a Hausdorff space, \((X, \mathcal{M}, \mu)\) a measure space, with \( \mu C < \infty \) for all compact \( C \in \mathcal{M} \) and \( \mu \) inner regular on \( S \in \mathcal{M} \) whenever \( \mu S < \infty \). Let

\[
\lambda S = \sup \{ \mu C : C \text{ compact}, C \subseteq S \},
\]

\[
\nu S = \inf \{ \mu U : U \text{ open}, U \in \mathcal{M}, U \supseteq S \},
\]

for all \( S \in \mathcal{M} \). Then

1. \( \lambda S = \mu S = \nu S \) whenever \( \nu S < \infty \),
2. \( \lambda \) and \( \nu \) are measures.

**Proof.** (1) Suppose \( \nu S < \infty \). Let \( V \) be a \( G_\delta \) set such that \( S \subseteq V \) and \( \nu S = \mu V \). If \( C \) is a compact member of \( \mathcal{M} \) with \( C \subseteq V - S \), then \( S \subseteq V - C \subseteq V \). Now \( V - C \) is a \( G_\delta \) set so \( \mu (V - C) = \nu S = \mu V \), thus \( \mu C = 0 \). Hence \( \mu (V - S) = 0 \), and \( \mu S = \mu V = \nu S \); \( \lambda S = \mu S \) is obvious since \( \mu S < \infty \).

(2) Suppose \( \{ S_j : j = 1, 2, \ldots \} \subseteq \mathcal{M} \) and \( S_j \cap S_k = \emptyset \) \((j \neq k)\). Let \( S = \bigcup S_j \), let \( C \) be a \( \sigma \)-compact subset of \( S \) such that \( \lambda S = \mu C \) and (for each \( j \)) let \( C_j \) be a \( \sigma \)-compact subset of \( S_j \) with \( \mu C_j = \lambda S_j \). Then

\[
\lambda S = \mu C = \mu (\bigcup C \cap S_j) = \sum \mu (C \cap S_j) \leq \sum \lambda S_j
\]

and

\[
\lambda S \geq \mu (\bigcup C_j) = \sum \mu C_j = \sum \lambda S_j,
\]

hence \( \lambda \) is a measure. Clearly (by (1)),

\[
\nu S = \mu S = \sum \mu S_j = \sum \nu S_j
\]

if \( \nu S < \infty \). If \( \nu S = \infty \), take \( U_j \) a \( G_\delta \) set such that \( U_j \supseteq S_j \) and \( \nu S_j = \mu U_j \) \((j = 1, 2, \ldots)\). Then

\[
\sum \nu S_j = \sum \mu U_j \geq \mu (\bigcup U_j) \geq \nu S = \infty,
\]

thus \( \nu \) is a measure.

**Lemma 2.** Let \( G \) be a locally compact group, \( H \) an open subgroup of \( G \). Then \( M(H) = M(G) \cap P(H) \).

**Proof.** Let \( \nu \) be a Haar measure on \( G \) and \( \nu_1 \) a Haar measure on \( H \). Both \( \nu \) and \( \nu_1 \) are unique to within a multiplicative constant; furthermore, if \( U \) is an open subset of \( H \) with compact closure, then \( 0 < \nu U < \infty \) and \( 0 < \nu_1 U < \infty \). Thus we may assume that \( \nu U = \nu_1 U \); but then the Carathéodory outer measures associated with \( \nu \) and \( \nu_1 \), respectively, are equal on \( P(H) \). It follows that the \( \nu \)-measurable and \( \nu_1 \)-measurable subsets of \( H \) coincide, which is to say that \( M(H) = M(G) \cap P(H) \).
Note. The proof of Lemma 2 contains the information that there is a one-to-one correspondence between the Haar measures on $G$ and $H$, respectively, given by $v \leftrightarrow v_1 = v|\mathcal{M}(H)$. One by-product of Theorem 1 will be a method of computing $v$, given $v_1$.

**Theorem 1.** Let $G$ be a nondiscrete locally compact group, let $H$ be an open subgroup of $G$, and $v_1$ a left Haar measure on $H$. For $S \subseteq \mathcal{M}(G)$, define

$$
\mu_S = \sum \{ v_1(xS \cap H) : xH \in G/H \},
$$

$$
\nu_S = \inf \{ \mu U : U \text{ open}, U \supseteq S \}.
$$

Then

1. $\mu$ is a well-defined left-invariant measure on $\mathcal{M}(G)$.
2. $\nu$ is a Haar measure for $G$.
3. $\mu$ and $\nu$ are both extensions of $v_1$, and $\mu$ and $\nu$ agree on open sets and $\sigma$-finite sets.
4. If $H$ is not $\sigma$-compact, $\mu$ fails to be inner regular on some closed subsets of $G$.
5. If $G/H$ is uncountable, $\mu$ fails to be outer regular on some closed subsets of $G$.

**Proof.** (1) We know that if $S \in \mathcal{M}(G)$, then $xS \in \mathcal{M}(G)$ and therefore $xS \cap H \in \mathcal{M}(G) \cap \mathcal{P}(H) = \mathcal{M}(H)$ for all $x \in G$. Further, if $xH = yH$, then

$$
v_1(xS \cap H) = v_1(yx^{-1}xS \cap H) = v_1(yS \cap H),
$$

since $v_1$ is left invariant. Thus $\mu$ is well defined; it is clearly left-invariant. To show that $\mu$ is a measure, suppose $\{ S_j \} \subseteq \mathcal{M}(G)$, $S_j \cap S_k = \emptyset$ ($j \neq k$); then

$$
\mu(\bigcup_{j=1}^\infty S_j) = \sum v_1(\bigcup_{j=1}^\infty xS_j \cap H) = \sum \left( \sum_{j=1}^\infty v_1(xS_j \cap H) \right)
$$

(by standard arguments; either both double summations have uncountably many nonzero terms or the $l_1$ version of Fubini's theorem applies).

(2) Let $S \in \mathcal{M}(G)$, with $S$ open or $\mu S < \infty$. There is a countable set $\{ x_j \}$ such that $\mu S = \sum_{j=1}^\infty v_1(x_j S \cap H)$. For each $j$, there is a $\sigma$-compact set $C_j$ such that $C_j \subseteq x_j S \cap H$ and $v_1 C_j = v_1(x_j S \cap H)$. Thus

$$
\mu S = \sum v_1 C_j = \sum \mu(x_j^{-1} C_j) = \mu(\bigcup_{j=1}^\infty x_j^{-1} C_j) \leq \lambda S \leq \mu S,
$$
where $\lambda$ is as in Lemma 1, since $\bigcup x \in C_j$ is a $\sigma$-compact subset of $S$. Now Lemma 1 applies, so that $\nu$ is a regular measure defined on $M(G)$; it is obvious that $\nu$ has the other properties required of a Haar measure.

(3) This statement is obvious.

(4) If $H$ is not $\sigma$-compact, then (16.14) of [2] shows that there is a closed subset of $H$ on which $\nu$ (and therefore $\mu$) is not inner regular.

(5) If $G/H$ is uncountable, then an argument easily derived from the proof of (16.14) (op. cit.) shows that there is a closed subset $F$ of $G$ such that $\mu F = 0$ and $\nu F = \infty$; thus $\mu U = \infty$ for any neighborhood $U$ of $F$ and $\mu$ is not outer regular on $F$.

Notes on Theorem 1. I. Statements (4) and (5) each imply that $G$ is not $\sigma$-compact. Theorems 2 and 3, below, state conditions under which (4) and (5) can be true for the same subgroup $H$.

II. $\lambda$ is the inner-regular “Haar measure” described in [1, Theorem 1] (and, from a different point of view, in [4, II.1])—or, more properly, the extension of this (weakly) Borel measure to $M(G)$.

III. For each $S$ in $M(G)$, one of the following statements must always be true:

(a) $\lambda S = \mu S = \nu S$ ($\mu$ is outer regular and inner regular on $S$).
(b) $\lambda S = \mu S < \nu S = \infty$ ($\mu$ is not outer regular on $S$).
(c) $\lambda S < \mu S = \nu S = \infty$ ($\mu$ is not inner regular on $S$).

IV. If $\nu_1$ were a right Haar measure, one could proceed in the same manner to obtain right-invariant measures on $M(G)$ with the desired properties, except that right cosets of $H$ and right translates of sets would play the rôle given to left cosets and left translates in Theorem 1.

3. Lemma 3. Let $G$ be an uncountable Abelian group. Then $G$ contains a subgroup $K$ such that $|K| = |G/K| = |G|$.

Proof. Let $r = |G|$.

Case 1. Suppose $r = r_0(G)$, the torsion-free rank of $G$. Then there exists a maximal independent torsion-free subset $X$ of $G$ such that $|X| = r$. Let $K = \{x^2 : x \in X\}$. As in [4, II.8], $|K| = |G/K| = r$.

Case 2. Suppose $G$ is torsion. Since $G$ is uncountable, it must have a subgroup $G_1$ of bounded order such that $|G_1| = r$. By (A.25) of [2], $G_1$ is the direct sum of cyclic groups; thus $G_1 = [Y]$ where $Y$ is an independent set and $|Y| = r$. Let $Y = Y_1 \cup Y_2$ where $Y_1 \cap Y_2 = \emptyset$ and $|Y_1| = |Y_2| = r$; let $K = [Y_1]$. Then $|K| = r \geq |G/K| \geq |G_1/K| \geq |Y_2| = r$.

Case 3. Suppose $r > r_0(G)$. Let $X$ be a maximal torsion-free independent set; let $G' = G/[X]$. $G'$ is a torsion group and $|G'| = r$, so by
Case 2 $G'$ has a subgroup $K'$ such that $|K'| = |G'/K'| = r$; let $K$ be the subgroup of $G$ such that $K' = K/\{x\}$. Clearly $|K| = r$, and by the Third Isomorphism Theorem $|G/K| = |G'/K'| = r$.

**Theorem 2.** Let $G$ be a locally compact Abelian group which is not $\sigma$-compact. Then $G$ has an open subgroup $H$ such that $H$ is not $\sigma$-compact and $G/H$ is uncountable.

**Proof.** Let $U$ be an open $\sigma$-compact subgroup of $G$; then $G' = G/U$ is uncountable and by Lemma 3 has a subgroup $K'$ such that $|K'| = |G'/K'| = |G'|$. Let $H$ be the subgroup of $G$ such that $K' = H/U$. Then $H$ is not $\sigma$-compact since $H/U$ is a cover of $H$ by uncountably many pairwise disjoint open sets. Also, $|G/H| = |G'/K'| = |G'|$.

**Theorem 3.** Let $G$ be a locally compact group which is not the union of fewer than $\aleph_2$ compact sets. Then $G$ has an open subgroup $H$ such that $H$ is not $\sigma$-compact and $G/H$ is uncountable.

**Proof.** Let $U$ be a $\sigma$-compact open subgroup of $G$ and let $H$ be a subgroup of $G$ generated by a collection of $\aleph_1$ cosets of $U$. $H$ has the desired properties.

4. **Examples (the group $R_d \times R$).** Let $G$ be the group $R_d \times R$, where $R_d$ is the discrete reals and $R$ is the reals with the usual topology. Let $\lambda_0$ be Lebesgue measure on $R$, and for $r \in R_d$, let $\lambda_r(S) = \lambda_0(\{x: (r, x) \in S\})$. Define

$$\lambda S = \sum \{\lambda_r(S): r \in R_d\}.$$

**Case 1.** (From [3, §12.58]). Let $H = \{0\} \times R$ and $\nu_1 = \lambda |M(H)$ be Lebesgue measure on $\{0\} \times R$. Here $H$ is $\sigma$-compact and $G/H$ is uncountable, being isomorphic to $R_d$. We have $\lambda = \mu$, and $\mu F_1 = 0$, $\nu F_1 = \infty$, where $F_1 = R_d \times \{0\}$.

**Case 2.** Let $K$ be the subgroup of $R_d$ generated by a Hamel basis over $Q$; let $H = K \times R$. In this case, $H$ is not $\sigma$-compact and $G/H$ is uncountable. For $S \in M(G)$, we have

$$\nu S = \inf \{\lambda U: U \text{ open, } U \supset S\},$$

$$\nu_1 = \nu |M(H),$$

as natural choices for Haar measures. Here,

$$\mu S = \sum \{\nu(S \cap (\{r\} \times R)): r \in K_2\}$$

where $K_2$ is a subgroup of $R_d$ such that $R_d = K_2 \oplus K$. Let $F_2 = K_2 \times \{0\}$; then $\lambda F_1 = 0$, $\mu F_1 = \nu F_1 = \infty$ and $\lambda F_2 = \mu F_2 = 0$, $\nu F_2 = \infty$. 

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