IRREGULAR INVARIANT MEASURES RELATED TO HAAR MEASURE

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ABSTRACT. Let G be a locally compact nondiscrete group, and let ν_1 be a Haar measure on an open subgroup of G. It is not hard to show that ν_1 must be the restriction of a Haar measure ν on all of G. Here we show that there exists a translation invariant measure μ (found by extending ν_1 to the cosets of H in a natural way) which agrees with ν on, for example, (ν) σ-finite sets, open sets, and subsets of H. Although ν can be computed from μ in a relatively simple manner, the two measures are not equal in general. In fact, there is an extreme case, namely when H is not σ-compact and has uncountably many cosets, in which μ fails very badly to be regular—there are closed sets on which μ is not inner regular and (other) closed sets on which μ is not outer regular. One condition sufficient for this extreme case to be possible is when G is Abelian and not σ-compact.

1. Definitions and notation. Let μ be a (nonnegative, countably additive) measure defined on a σ-algebra $\boldsymbol{M}$ of subsets of a topological space X. If S is in $\boldsymbol{M}$, we say that μ is *inner regular* on S if

$$\mu S = \sup\{\mu C : C \in \boldsymbol{M},\ C \text{ compact},\ C \subset S\}.$$

We say that μ is *outer regular* on S if

$$\mu S = \inf\{\mu U : U \in \boldsymbol{M},\ U \text{ open},\ U \supset S\}.$$

Following [2, 11.34], we say that μ is *regular* if it is outer regular on every set in $\boldsymbol{M}$ and inner regular on every open set in $\boldsymbol{M}$, and if every compact set in $\boldsymbol{M}$ has finite measure.

By *Haar measure* on a locally compact group G, we mean a left Haar measure as defined in, e.g., [2]; that is to say, a left-translation invariant, regular, nondegenerate measure on a σ-algebra $\boldsymbol{M}(G)$ of subsets of G. $\boldsymbol{M}(G)$ contains all the closed subsets of G and consists of all the sets which are measurable with respect to the Carathéodory outer measure associated with the measure.

If H is a subgroup of G (not necessarily normal), G/H denotes the space of left cosets of H in G. If S is a set, $\boldsymbol{P}(S)$ denotes the collection of all subsets of S; $|S|$ denotes the cardinality of S.

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2. **Lemma** 1. *Let X be a Hausdorff space, (X, M, μ) a measure space, with $\mu C < \infty$ for all compact $C \in M$ and μ inner regular on $S \in M$ whenever $\mu S < \infty$. Let*

$$\lambda S = \sup\{\mu C:\ C \text{ compact},\ C \in M,\ C \subset S\},$$
$$\nu S = \inf\{\mu U:\ U \text{ open},\ U \in M,\ U \supset S\},$$

for all $S \in M$. Then

(1) *$\lambda S = \mu S = \nu S$ whenever $\nu S < \infty$,*

(2) *λ and ν are measures.*

Proof. (1) Suppose $\nu S < \infty$. Let V be a G_δ set such that $S \subset V$ and $\nu S = \mu V$. If C is a compact member of M with $C \subset V - S$, then $S \subset V - C \subset V$. Now $V - C$ is a G_δ set so $\mu(V-C) = \nu S = \mu V$, thus $\mu C = 0$. Hence $\mu(V-S) = 0$, and $\mu S = \mu V = \nu S$; $\lambda S = \mu S$ is obvious since $\mu S < \infty$.

(2) Suppose $\{S_j: j = 1, 2, \cdots\} \subset M$ and $S_j \cap S_k = \emptyset$ $(j \neq k)$. Let $S = \bigcup S_j$, let C be a σ-compact subset of S such that $\lambda S = \mu C$ and (for each j) let C_j be a σ-compact subset of S_j with $\mu C_j = \lambda S_j$. Then

$$\lambda S = \mu C = \mu(\bigcup C \cap S_j) = \sum \mu(C \cap S_j) \leqq \sum \lambda S_j$$

and

$$\lambda S \geqq \mu(\bigcup C_j) = \sum \mu C_j = \sum \lambda S_j,$$

hence λ is a measure. Clearly (by (1)),

$$\nu S = \mu S = \sum \mu S_j = \sum V S_j$$

if $\nu S < \infty$. If $\nu S = \infty$, take U_j a G_δ set such that $U_j \supset S_j$ and $\nu S_j = \mu U_j$ $(j = 1, 2, \cdots)$. Then

$$\sum \nu S_j = \sum \mu U_j \geqq \mu(\bigcup U_j) \geqq \nu S = \infty,$$

thus ν is a measure.

Lemma 2. *Let G be a locally compact group, H an open subgroup of G. Then $M(H) = M(G) \cap P(H)$.*

Proof. Let ν be a Haar measure on G and ν_1 a Haar measure on H. Both ν and ν_1 are unique to within a multiplicative constant; further, if U is an open subset of H with compact closure, then $0 < \nu U < \infty$ and $0 < \nu_1 U < \infty$. Thus we may assume that $\nu U = \nu_1 U$; but then the Carathéodory outer measures associated with ν and ν_1, respectively, are equal on $P(H)$. It follows that the ν-measurable and ν_1-measurable subsets of H coincide, which is to say that $M(H) = M(G) \cap P(H)$.

Note. The proof of Lemma 2 contains the information that there is a one-to-one correspondence between the Haar measures on G and H, respectively, given by $\nu \leftrightarrow \nu_1 = \nu | M(H)$. One by-product of Theorem 1 will be a method of computing ν, given ν_1.

THEOREM 1. *Let G be a nondiscrete locally compact group, let H be an open subgroup of G, and ν_1 a left Haar measure on H. For $S \in \boldsymbol{M}(G)$, define*

$$\mu S = \sum \{\nu_1(xS \cap H) : xH \in G/H\},$$
$$\nu S = \inf\{\mu U : U \text{ open}, U \supset S\}.$$

Then

(1) *μ is a well-defined left-invariant measure on $\boldsymbol{M}(G)$.*

(2) *ν is a Haar measure for G.*

(3) *μ and ν are both extensions of ν_1 and μ and ν agree on open sets and (ν) σ-finite sets.*

(4) *If H is not σ-compact, μ fails to be inner regular on some closed subsets of G.*

(5) *If G/H is uncountable, μ fails to be outer regular on some closed subsets of G.*

PROOF. (1) We know that if $S \in \boldsymbol{M}(G)$, then $xS \in \boldsymbol{M}(G)$ and therefore $xS \cap H \in \boldsymbol{M}(G) \cap \boldsymbol{P}(H) = \boldsymbol{M}(H)$ for all $x \in G$. Further, if $xH = yH$, then

$$\nu_1(xS \cap H) = \nu_1(yx^{-1}xS \cap H) = \nu_1(yS \cap H),$$

since ν_1 is left invariant. Thus μ is well defined; it is clearly left-invariant. To show that μ is a measure, suppose $\{S_j\} \subset \boldsymbol{M}(G)$, $S_j \cap S_k = \emptyset$ $(j \neq k)$; then

$$\mu(\cup S_j) = \sum \nu_1(\cup xS_j \cap H) = \sum \left(\sum_{j=1}^{\infty} \nu_1(xS_j \cap H) \right)$$
$$= \sum_{j=1}^{\infty} \left(\sum \nu_1(xS_j \cap H) \right) = \sum_{j=1}^{\infty} \mu S_j$$

(by standard arguments; either both double summations have uncountably many nonzero terms or the l_1 version of Fubini's theorem applies).

(2) Let $S \in \boldsymbol{M}(G)$, with S open or $\mu S < \infty$. There is a countable set $\{x_j\}$ such that $\mu S = \sum_{j=1}^{\infty} \nu_1(x_j S \cap H)$. For each j, there is a σ-compact set C_j such that $C_j \subset x_j S \cap H$ and $\nu_1 C_j = \nu_1(x_j S \cap H)$. Thus

$$\mu S = \sum \nu_1 C_j = \sum \mu(x_j^{-1} C_j) = \mu(\cup x_j^{-1} C_j) \leqq \lambda S \leqq \mu S,$$

where λ is as in Lemma 1, since $\bigcup x_j^{-1}C_j$ is a σ-compact subset of S. Now Lemma 1 applies, so that ν is a regular measure defined on $\boldsymbol{M}(G)$; it is obvious that ν has the other properties required of a Haar measure.

(3) This statement is obvious.

(4) If H is not σ-compact, then (16.14) of [2] shows that there is a closed subset of H on which ν (and therefore μ) is not inner regular.

(5) If G/H is uncountable, then an argument easily derived from the proof of (16.14) (op. cit.) shows that there is a closed subset F of G such that $\mu F=0$ and $\nu F=\infty$; thus $\mu U=\infty$ for any neighborhood U of F and μ is not outer regular on F.

NOTES ON THEOREM 1. I. Statements (4) and (5) each imply that G is not σ-compact. Theorems 2 and 3, below, state conditions under which (4) and (5) can be true for the same subgroup H.

II. λ is the inner-regular "Haar measure" described in [**1**, Theorem 1] (and, from a different point of view, in [**4**, II.1])—or, more properly, the extension of this (weakly) Borel measure to $\boldsymbol{M}(G)$.

III. For each S in $\boldsymbol{M}(G)$, one of the following statements must always be true:

(a) $\lambda S=\mu S=\nu S$ (μ is outer regular and inner regular on S).

(b) $\lambda S=\mu S<\nu S=\infty$ (μ is not outer regular on S).

(c) $\lambda S<\mu S=\nu S=\infty$ (μ is not inner regular on S).

IV. If ν_1 were a *right* Haar measure, one could proceed in the same manner to obtain right-invariant measures on $\boldsymbol{M}(G)$ with the desired properties, except that right cosets of H and right translates of sets would play the rôle given to left cosets and left translates in Theorem 1.

3. **Lemma** 3. *Let G be an uncountable Abelian group. Then G contains a subgroup K such that $|K|=|G/K|=|G|$.*

PROOF. Let $r=|G|$.

Case 1. Suppose $r=r_0(G)$, the torsion-free rank of G. Then there exists a maximal independent torsion-free subset X of G such that $|X|=r$. Let $K=[\{x^2: x\in X\}]$. As in [**4**, II.8], $|K|=|G/K|=r$.

Case 2. Suppose G is torsion. Since G is uncountable, it must have a subgroup G_1 of bounded order such that $|G_1|=r$. By (A.25) of [2], G_1 is the direct sum of cyclic groups; thus $G_1=[Y]$ where Y is an independent set and $|Y|=r$. Let $Y=Y_1\cup Y_2$ where $Y_1\cap Y_2=\emptyset$ and $|Y_1|=|Y_2|=r$; let $K=[Y_1]$. Then $|K|=r\geqq|G/K|\geqq|G_1/K|\geqq|Y_2|=r$.

Case 3. Suppose $r>r_0(G)$. Let X be a maximal torsion-free independent set; let $G'=G/[X]$. G' is a torsion group and $|G'|=r$, so by

Case 2 G' has a subgroup K' such that $|K'| = |G'/K'| = r$; let K be the subgroup of G such that $K' = K/[X]$. Clearly $|K| = r$, and by the Third Isomorphism Theorem $|G/K| = |G'/K'| = r$.

THEOREM 2. *Let G be a locally compact Abelian group which is not σ-compact. Then G has an open subgroup H such that H is not σ-compact and G/H is uncountable.*

PROOF. Let U be an open σ-compact subgroup of G; then $G' = G/U$ is uncountable and by Lemma 3 has a subgroup K' such that $|K'| = |G'/K'| = |G'|$. Let H be the subgroup of G such that $K' = H/U$. Then H is not σ-compact since H/U is a cover of H by uncountably many pairwise disjoint open sets. Also, $|G/H| = |G'/K'| = |G'|$.

THEOREM 3. *Let G be a locally compact group which is not the union of fewer than $\aleph_2$ compact sets. Then G has an open subgroup H such that H is not σ-compact and G/H is uncountable.*

PROOF. Let U be a σ-compact open subgroup of G and let H be a subgroup of G generated by a collection of $\aleph_1$ cosets of U. H has the desired properties.

4. **Examples (the group $R_d \times R$).** Let G be the group $R_d \times R$, where R_d is the discrete reals and R is the reals with the usual topology. Let λ_0 be Lebesgue measure on R, and for $r \in R_d$, let $\lambda_r(S) = \lambda_0(\{x: (r, x) \in S\})$. Define

$$\lambda S = \sum \{\lambda_r(S): r \in R_d\}.$$

Case 1. (*From* [3, §12.58]). Let $H = \{0\} \times R$ and $\nu_1 = \lambda | M(H)$ = Lebesgue measure on $\{0\} \times R$. Here H is σ-compact and G/H is uncountable, being isomorphic to R_d. We have $\lambda = \mu$, and $\mu F_1 = 0$, $\nu F_1 = \infty$, where $F_1 = R_d \times \{0\}$.

Case 2. Let K be the subgroup of R_d generated by a Hamel basis over Q; let $H = K \times R$. In this case, H is not σ-compact and G/H is uncountable. For $S \in M(G)$, we have

$$\nu S = \inf\{\lambda U: U \text{ open}, U \supset S\},$$
$$\nu_1 = \nu | M(H),$$

as natural choices for Haar measures. Here,

$$\mu S = \sum \{\nu(S \cap (\{r\} \times R)): r \in K_2\}$$

where K_2 is a subgroup of R_d such that $R_d = K_2 \oplus K$. Let $F_2 = K_2 \times \{0\}$; then $\lambda F_1 = 0$, $\mu F_1 = \nu F_1 = \infty$ and $\lambda F_2 = \mu F_2 = 0$, $\nu F_2 = \infty$.

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