IRREDUCIBLE MARKOV OPERATORS ON $C(S)$

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1. Let S be a compact Hausdorff space, $C(S)$ the Banach space of all real-valued continuous functions on S with the supremum norm. A linear operator T on $C(S)$ is said to be a *Markov operator* if $T(1)=1$ and if $f \geqq 0$ implies $Tf \geqq 0$. It is known [5] that such an operator is of the form

$$(Tf)(x) = \int f(y)P(x, dy), \qquad x \in C(S),$$

where, for each $x \in S$, $P(x, \cdot)$ is a regular probability measure on the σ-field Σ of Borel subsets of S, and where the map $x \to P(x, \cdot)$ is continuous relative to the weak* topology on $C^*(S)$. The operator T is called *irreducible* if for each nonnegative but somewhere positive $f \in C(S)$ and for each $x \in S$ there is an n such that $(T^n f)(x) > 0$. This is the case if and only if for each open set U and each x there is an n for which $P^n(x, U) > 0$. If T is irreducible, then $Tf = f$ and $f \in C(S)$ if and only if f is constant on S (Theorem 2.4 of [4]). A nonempty $E \in \Sigma$ is said to be *stochastically closed* if $P(x, E) = 1$ for each $x \in E$. If T is irreducible then no proper closed subset of S is stochastically closed. We say that T is *strongly* (*weakly*) *almost periodic* if for each $f \in C(S)$ the set $\{T^n f: n = 1, 2, \cdots\}$ has a cluster point relative to the strong (weak) topology on $C(S)$ (see [1], [5]).

The purpose of this note is to prove the following

THEOREM. *Suppose T is an irreducible Markov operator on $C(S)$, where S is compact Hausdorff. Let $f \in C(S)$. If*

$$\lim_n (T^n f)(x) = 0, \qquad x \in S,$$

then $\|T^n f\| \to 0$.

We give the proof in §3. Rosenblatt, sharpening results of deLeeuw and Glicksberg [1], has shown [5, p. 217], that if T is irreducible and weakly almost periodic then

$$C(S) = C_p \oplus C_0,$$

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where C_p is the closed linear span of the eigenfunctions corresponding to the unimodular eigenvalues of T and where

$$C_0 = \{f: (T^n f)(x) \to 0, \ \forall x \in S\}.$$

Since $\{T^n f\}$ is conditionally compact in the strong topology for each $f \in C_0$, our theorem has the following

COROLLARY. *An irreducible weakly almost periodic Markov operator on $C(S)$ is strongly almost periodic.*

2. In this section we assume that S is compact metric with metric d. Let $\{f_n\} \subset C(S)$, $\|f_n\| \leqq k < \infty$ for all n. For each $x \in S$ we define

$$O(x) = \lim_m \limsup_n (\sup\{ |f_n(y)| : d(x, y) \leqq 1/m\}).$$

It is not hard to see that if $n_k \to \infty$ and $x_k \to x$ then lim sup (sup $|f_{n_k}(x_k)|$) $\leqq O(x)$, and also that there is an $n_k \to \infty$ and $x_k \to x$ with $|f_{n_k}(x_k)| \to O(x)$. We list three of the properties of $O(x)$ as a function on S, leaving the rather routine verifications of the first two to the reader.

A. *$O(x)$ is upper semicontinuous.*
In particular, $O(x)$ assumes its least upper bound α.
B. $\|(|f_n| \vee \alpha) - \alpha\| \to 0$ *as* $n \to \infty$.
C. *If $f_n(x) \to 0$ for each $x \in S$, $\{x: O(x) > 0\}$ is a set of the first category.*

PROOF of C. Assume that $f_n \to 0$ pointwise on S. For each $j = 1, 2, \cdots$, let

$$O_j = \{x: O(x) \geqq 1/j\}.$$

Each O_j is closed by virtue of (A). Fix j and suppose that O_j has nonempty interior U. Then there is a closed $V \subset U$ with nonempty interior. Fix each $j = 1, 2, \cdots$, let

$$S_i = \{x: x \in V \text{ and } |f_k(x)| \leqq 1/(2j) \text{ for all } k \geqq i\}.$$

Each S_i is a closed subset of V. Since $f_n \to 0$ on S the S_i's cover V. But V itself is a complete metric space, so at least one of the S_i's must have a nonempty interior W. On the one hand $W \subset O_j = \{x: O(x) \geqq 1/j\}$, and on the other hand it is clear that $W \subset \{x: O(x) \leqq 1/(2j)\}$. This is a contradiction. It follows that each of the O'_j's has empty interior. Since $\{x: O(x) > 0\}$ is the union of the O_j's, it is a set of the first category.

3. This section is devoted to the proof of the theorem of §1. Assume that $T^n f \to 0$ pointwise. We first consider the case where S is

compact metric. Let $\{T^n f\}$ be the sequence $\{f_n\}$ of the preceding section, with $\alpha = \sup_x O(x)$. Suppose $\alpha > 0$. Let $E = \{x: O(x) = \alpha\}$. Since $E = \{x: O(x) \geqq \alpha\}$, E is closed and $\neq \emptyset$ by virtue of (A). Claim: E is *stochastically closed.*

This is trivial if $E = S$. Otherwise, let $x \in E$. There are sequences $x_k \to x$ and $n_k \to \infty$ with $|T^{n_k+1} f(x_k)| \to \alpha$. Now suppose $P(x, E^c) > 0$. It is easy to show that there is a closed set $F \subset E^c$ and a number $p > 0$ such that $P(x_k, F) \geqq p$ for all sufficiently large k. Let $\beta = \sup_{x \in F} O(x)$. Since F is closed, (A) yields a $z \in F$ with $O(z) = \beta$; since $F \subset E^c$, $\beta < \alpha$. Let $2\epsilon = \alpha - \beta$. Suppose $|(T^{n_k} f)(y)| \vee (\alpha - \epsilon) \to \alpha - \epsilon$ as $k \to \infty$ uniformly for $y \in F$. We have

$$(T^{n_k+1} f)(x_k) = \int_F (T^{n_k} f)(y) P(x_k, dy) + \int_{F^c} (T^{n_k} f(y)) P(x_k, dy).$$

However, $|T^n f| \vee \alpha \to \alpha$ uniformly on S, hence on F^c (by virtue of (B)). Since $P(x_k, F) \geqq p$ for all sufficiently large k, it would follow that

$$\alpha = \lim |T^{n_k+1} f(x_k)| \leqq (\alpha - \epsilon)p + \alpha(1 - p) = \alpha - p\epsilon < \alpha,$$

which is, of course, a contradiction. Thus we can find a subsequence $\{k_j\}$ and a sequence $y_j \in F$ such that $|T^{n_{k_j}} f(y_j)| > \alpha - \epsilon$. We may assume without loss of generality that $\{y_j\}$ converges, its limit y being necessarily in F. But then $O(y) \geqq \alpha - \epsilon > \alpha - 2\epsilon = \beta$, which is an impossibility since $y \in F$ and $\sup_{x \in F} O(x) = \beta$. This shows that $P(x, E^c) = 0$. Since x is arbitrary, this establishes that E is stochastically closed.

Since E is both topologically and stochastically closed, $E = S$ by irreducibility. But this contradicts (C). Thus $\alpha = 0$, whence $\|T^n f\| \to 0$ by virtue of (B).

Now suppose S is not metric. Let $A_1 = \{f\}$. For each $n = 1, 2, \cdots$, let

$$A_{2n} = A_{2n-1} \cup \{Tg: g \in A_{2n-1}\},$$

and let A_{2n+1} be the union of A_{2n} with the class of all finite products of members of A_{2n}. Let $A_\infty = \{1\} \cup \bigcup_{n=1}^\infty A_n$. Then $A_\infty A_\infty \subset A_\infty$. Let A_0 be the collection of all finite rational linear combinations of members of A_∞. Then $A_0 A_0 \subset A_0$, $TA_0 \subset A_0$, and A_0 is closed under rational linear combinations, Let A be the uniform closure of A. Then A is a closed subalgebra of $C(S)$ containing $T^n f$ for each $n = 1, 2, \cdots$, and $TA \subset A$. The level sets of A from a set $\tilde{S}$ which is compact Hausdorff relative to its quotient topology. To each $f \in C(S)$ which is constant on the level sets of A there corresponds an $\tilde{f} \in C(\tilde{S})$, and vice versa.

To A corresponds a closed subalgebra $\tilde{A}$ of $C(\tilde{S})$ which separates points and contains 1, so $\tilde{A}=C(\tilde{S})$ by the Stone-Weierstrass Theorem. Since A is separable, so is $C(\tilde{S})$, hence $\tilde{S}$ is metrizable. Since $TA\subset A$, the restriction of T to A lifts to a Markov operator $\tilde{T}$ on $C(\tilde{S})$ for which $\tilde{T}^n\tilde{f}=T^nf$. The irreducibility of $\tilde{T}$ follows from that of T. Now, $T^nf\rightarrow 0$ pointwise$\Rightarrow\tilde{T}^n\tilde{f}\rightarrow 0$ pointwise$\Rightarrow\|\tilde{T}^n\tilde{f}\|\rightarrow 0$ by the proof for the metric case, so $\|T^nf\|=\|\tilde{T}^n\tilde{f}\|\rightarrow 0$. This completes the proof of the theorem.

4. Suppose S is the one point compactification of the integers, the latter having the discrete topology. As usual, ∞ denotes the point at infinity. Suppose $P(j,\{j+1\})=1, j=0,\pm 1,\cdots$, and that $P(\infty,\{\infty\})=1$. Then $(T^nf)(x)$ converges pointwise to $f(\infty)$ for each $f\in C(S)$, so T is weakly almost periodic. If $f(0)=1$ and $f(x)=0$ for $x\in S-\{0\}$, then $f(\infty)=0$, but $(T^nf)(-n)=1$. Thus T is not strongly almost periodic. This shows that the condition of irreducibility is essential for the truth of corollary.

Suppose that S is locally compact but not compact, and that $C(S)$ is the space of all bounded continuous functions. Then even if T is irreducible and $T^nf\rightarrow 0$ pointwise, $\{T^nf\}$ need not converge uniformly. To see this, let S be the nonnegative integers with the discrete topology. Then if T is a Markov operator on $C(S)$, there is a matrix P on $S\times S$ such that $(T^nf)(i)=\Sigma_{j=0}^{\infty} p^{(n)}(i,j)f(j)$, $i\in S$, where $p^{(n)}(i,j)$ is the (i,j)th entry of P^n. Assume that P is irreducible, periodic, and positive recurrent (see [**3**, Chapter 13], for terminology). Then there is a probability measure μ on S such that $p^{(n)}(i,j)\rightarrow\mu(j)$ as $n\rightarrow\infty$. It follows easily that $T^nf\rightarrow\int f d\mu$ pointwise for each $f\in C(S)$. Were the convergence uniform for each such f, we would have in particular $p^{(n)}(i,0)\rightarrow\mu(0)$ uniformly in i. This is impossible if, for example, $p^{(n)}(n,0)=1$ but $p^{(m)}(n,0)=0$ for $m\leqq n$. For an example of such a P which is at the same time irreducible, aperiodic, and null recurrent, see Problem 6 on p. 376 of [**3**]. As a matter of fact, it is not difficult to see that if $p^{(n)}(i,0)\rightarrow\mu(0)>0$ uniformly in i then Doeblin's condition holds (see p. 192 of [**2**]). This is a very strong condition on P and, if it holds, the convergence is uniformly exponential; that is, there is a $0<\rho<1$ and a $K>0$ such that $|p^{(n)}(i,j)-\mu(j)|\leqq A\rho^n$ for all i,j and n (see [**2**, pp. 207–208]).

References

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