

## DISJOINTNESS AND WEAK MIXING OF MINIMAL SETS

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In [3, Problem G, p. 34] the following problem is proposed: characterize the class  $\mathfrak{D}^\perp$  of all flows which are disjoint from every distal flow. We show here that  $\mathfrak{D}^\perp$  consists precisely of the weakly mixing minimal flows. In arriving at this conclusion we make use of a result (Corollary to Theorem 1) which has been arrived at independently by Keynes and Robertson [4, Theorem 3.4, p. 366]. In the following all transformation groups will be assumed to have compact Hausdorff phase spaces. For any unexplained notation or terminology the reader is referred to [1] and [3]. This research is part of the author's doctoral dissertation prepared at Yale University under the guidance of Professor S. Kakutani, whose assistance the author gratefully acknowledges.

We say that a transformation group  $(X, T)$  is *weakly mixing* if given nonempty open subsets  $A, B, C, D$  of  $X$  there is  $t \in T$  such that  $At \cap C \neq \emptyset$  and  $Bt \cap D \neq \emptyset$ .

LEMMA. *Let  $(X, T)$  be a transformation group and suppose that  $T$  is abelian. Then  $(X, T)$  is weakly mixing if and only if given nonempty open subsets  $A$  and  $B$  of  $X$  there is  $t \in T$  such that  $At \cap A \neq \emptyset$  and  $At \cap B \neq \emptyset$ .*

PROOF. It is clear that if  $(X, T)$  is weakly mixing then the stated condition is satisfied. Suppose then that the condition holds and let nonempty open subsets  $A, B, C, D$  of  $X$  be given; we need to find  $t \in T$  such that  $At \cap C \neq \emptyset$  and  $Bt \cap D \neq \emptyset$ .

We may choose  $t_1 \in T$  such that  $E = At_1 \cap B \neq \emptyset$ ,  $t_2 \in T$  such that  $F = Et_2 \cap Ct_1 \neq \emptyset$ , and  $t_3 \in T$  such that  $Ft_3 \cap F \neq \emptyset$  and  $Ft_3 \cap D \neq \emptyset$ . Let  $t = t_2t_3$ . Then

$$\begin{aligned} (At \cap C)t_1 &= At_1t \cap Ct_1 \supseteq At_1t \cap Bt \cap Ct_1 = (At_1 \cap B)t \cap Ct_1 \\ &= E(t_2t_3) \cap Ct_1 \supseteq Ft_3 \cap Ct_1 \supseteq Ft_3 \cap F \neq \emptyset, \end{aligned}$$

so  $At \cap C \neq \emptyset$ ; and

$$At_1t \cap Bt \cap D = (At_1 \cap B)t \cap D = E(t_2t_3) \cap D \supseteq Ft_3 \cap D \neq \emptyset,$$

so  $Bt \cap D \neq \emptyset$ .

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If  $(X, T)$  is a transformation group, we will denote by  $RP$  and  $S_e$  the regionally proximal and equicontinuous structure relations of  $X$ , respectively [1, Definition 6, p. 261 and Definition 4, p. 260].

**THEOREM 1.** *If  $(X, T)$  is a weakly mixing transformation group, then  $RP = X \times X$ . If  $(X, T)$  is a minimal abelian transformation group for which  $S_e = X \times X$ , then  $(X, T)$  is weakly mixing.*

**PROOF.** The first statement in the theorem is easily verified. Suppose then that  $(X, T)$  is a minimal abelian transformation group for which  $S_e = X \times X$ . By a result of Veech [5, Theorem 1.1, p. 723], this implies that given  $x \in X$  and an open neighborhood  $U$  of  $x$ , the set  $\{xt_1t_2^{-1} : xt_1 \in U, xt_2 \in U\}$  is dense in  $X$ . Let nonempty open subsets  $A$  and  $B$  of  $X$  be given, and let  $x \in A$ . Then there are  $t_1, t_2 \in T$  such that  $xt_1 \in A$ ,  $xt_2 \in A$ , and  $xt_1t_2^{-1} \in B$ . Thus  $A(t_1t_2^{-1}) \cap B \neq \emptyset$ . Since  $xt_2 \in A$ , we have  $x \in At_2^{-1}$  and  $xt_1 \in At_2^{-1}t_1 = At_1t_2^{-1}$ , and hence  $xt_1 \in A(t_1t_2^{-1}) \cap A$ . The lemma now implies that  $(X, T)$  is weakly mixing.

**COROLLARY.** *For a minimal abelian transformation group  $(X, T)$ , with  $X$  compact metric, the following statements are equivalent:*

- (1)  $(X, T)$  is weakly mixing.
- (2)  $RP = X \times X$ .
- (3)  $S_e = X \times X$ .
- (4)  $(X, T)$  has no nontrivial distal homomorphic image.

**PROOF.** That (1), (2), and (3) are equivalent is direct from the theorem, remembering that  $RP \subseteq S_e$ . That (3) is equivalent to (4) is proved with the aid of [2, Theorem 2.4, p. 483].

By a *flow*  $(X, \phi)$  we will mean a compact metric space  $X$  together with a homeomorphism  $\phi: X \rightarrow X$ . We denote by  $\mathfrak{D}$  the class of all distal flows, by  $\mathfrak{M}$  the class of all minimal flows, and by  $\mathfrak{W}$  the class of all weakly mixing flows. If  $\mathfrak{C}$  is a class of flows,  $\mathfrak{C}^\perp$  will denote the class of all flows which are disjoint [3, Definition II.1, p. 24] from every flow in  $\mathfrak{C}$ .

**THEOREM 2.**  $\mathfrak{D}^\perp = \mathfrak{W} \cap \mathfrak{M}$ .

**PROOF.** It is proved in [3, Theorem II.3, p. 33] that  $(\mathfrak{D} \cap \mathfrak{M})^\perp \supseteq \mathfrak{W}$ , so we have  $(\mathfrak{D} \cap \mathfrak{M})^\perp \cap \mathfrak{M} \supseteq \mathfrak{W} \cap \mathfrak{M}$ . If  $(X, \phi) \in (\mathfrak{D} \cap \mathfrak{M})^\perp \cap \mathfrak{M}$ , then  $(X, \phi)$  can have no nontrivial distal homomorphic image, since it would not be disjoint from such an image. By the above Corollary, then,  $(X, \phi) \in \mathfrak{W} \cap \mathfrak{M}$ . Therefore  $(\mathfrak{D} \cap \mathfrak{M})^\perp \cap \mathfrak{M} = \mathfrak{W} \cap \mathfrak{M}$ .

It is a result of Furstenberg's [3, Theorem II.1, p. 32] that if two flows are disjoint then one of them must be minimal. Since  $\mathfrak{D}$  contains

flows which are not minimal, we have  $\mathcal{D}^\perp \cap \mathfrak{M} = \mathcal{D}^\perp$ . Since  $\mathcal{D} \cap \mathfrak{M} \subseteq \mathcal{D}$ , we have  $(\mathcal{D} \cap \mathfrak{M})^\perp \supseteq \mathcal{D}^\perp$ , and hence  $(\mathcal{D} \cap \mathfrak{M})^\perp \cap \mathfrak{M} \supseteq \mathcal{D}^\perp \cap \mathfrak{M} = \mathcal{D}^\perp$ . Thus  $\mathfrak{W} \cap \mathfrak{M} \supseteq \mathcal{D}^\perp$ .

Now let  $(X, \phi) \in \mathfrak{W} \cap \mathfrak{M}$  and  $(Y, \psi) \in \mathcal{D}$ ; we need to prove that  $(X, \phi)$  and  $(Y, \psi)$  are disjoint. This amounts to showing [3, Lemma II.1, p. 24] that if  $V$  is a closed invariant subset of the flow  $(X \times Y, \phi \times \psi)$  which projects onto  $X$  and onto  $Y$ , then  $V = X \times Y$ . Suppose that  $V$  is such a set. It is a well-known result of Ellis that,  $(Y, \psi)$  being distal,  $Y$  is the disjoint union of the minimal subsets  $Y_\alpha$  of  $(Y, \psi)$ . Thus it suffices to prove that  $V \supseteq X \times Y_\alpha$  for each  $\alpha$ . Now for each  $\alpha$ ,  $(Y_\alpha, \psi) \in \mathcal{D} \cap \mathfrak{M}$ , so by the first part of the proof  $(X, \phi)$  and  $(Y_\alpha, \psi)$  are disjoint; therefore  $(X \times Y_\alpha, \phi \times \psi)$  is minimal. Since  $V \cap (X \times Y_\alpha)$  is a nonempty closed invariant subset of  $(X \times Y_\alpha, \phi \times \psi)$ , we must have  $V \cap (X \times Y_\alpha) = X \times Y_\alpha$ . Therefore  $V = X \times Y$  and the proof is complete.

We remark that as an application of these results it can be proved that a minimal flow which is prime in the sense that it has no proper homomorphic images must be either a translation on a cyclic group of prime order or else weakly mixing. We conjecture that the corollary to Theorem 1 remains valid if the requirement that  $X$  be metrizable is dropped.

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