A NOTE ON CONVEX AND BAZILEVIČ FUNCTIONS
M. NUNOKAWA

1. Let \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) be regular, univalent in \(|z| < 1\) and map \(|z| < 1\) onto a domain which is starlike with respect to the origin. Then we call \( f(z) \) a starlike function. It is well known that a function \( f(z) \) is starlike in \(|z| < 1\) if and only if

\[
\text{Re} \left( \frac{zf''(z)}{f'(z)} \right) > 0 \quad \text{in} \quad |z| < 1.
\]

Let \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) be regular, univalent in \(|z| < 1\) and map \(|z| < 1\) onto a convex domain. Then we call \( f(z) \) a convex function. It is well known that a regular function \( f(z) \) is convex if and only if

\[
1 + \text{Re} \left( \frac{zf'''(z)}{f''(z)} \right) > 0 \quad \text{in} \quad |z| < 1.
\]

Every convex function is a starlike function [4].

Let \( L(r) \) denote the length of the closed curve \( C(r) \) which is the image of the circle \(|z| = r < 1\) under the mapping \( w = f(z) \) and \( A(r) \) the area enclosed by \( C(r) \).

Recently Thomas [6], [8] has shown that if \( f(z) \) is a starlike function, then

\[
L(r) \leq 2(\pi A(r))^{1/2} \left( 1 + \log \frac{1 + r}{1 - r} \right).
\]

In this note, for the convex functions we obtain a stronger result than (1).

**Theorem 1.** Let \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) be a convex function. Then we have

\[
L(r) = 0 \left( A(r) \log \frac{1}{1 - r} \right)^{1/2} \quad \text{as} \quad r \to 1.
\]

**Proof.** Since \( f(z) \) is a univalent function we can get

Received by the editors May 26, 1969.
A NOTE ON CONVEX AND BAZILEVIĆ FUNCTIONS

\[ L(r) = \int_0^{2\pi} |zf'(z)| \, d\theta \leq \int_0^{2\pi} \int_0^r \left| zf'''(z) + f'(z) \right| \, dp \, d\theta \]

\[ = \int_0^r \int_0^{2\pi} \left| f'(z) \left( 1 + \frac{zf'''(z)}{f'(z)} \right) \right| \, d\theta \, dp \]

\[ = \int_0^{r_1} \int_0^{2\pi} \left| f'(z) \left( 1 + \frac{zf'''(z)}{f'(z)} \right) \right| \, d\theta \, dp \]

\[ + \int_{r_1}^r \int_0^{2\pi} \left| f'(z) \left( 1 + \frac{zf'''(z)}{f'(z)} \right) \right| \, d\theta \, dp \]

where \( r_1 \) is a fixed constant and \( 0 < r_1 < r < 1 \).

Therefore we have

\[ L(r) \]

\[ \leq C + \left( \int_{r_1}^r \int_0^{2\pi} \rho \left| f'(z) \left( 1 + \frac{zf'''(z)}{f'(z)} \right) \right| ^2 \, d\theta \, dp \right)^{1/2} \]

\[ \leq C + \left( \int_0^{2\pi} \int_0^{2\pi} \frac{1}{\rho} \left| 1 + \frac{zf'''(z)}{f'(z)} \right| ^2 \, d\theta \, dp \right)^{1/2} \]

\[ \leq C + \left( \frac{A(r)}{r_1} \right)^{1/2} \left( \int_0^{2\pi} \int_0^{2\pi} \frac{1}{\rho} \left| 1 + \frac{zf'''(z)}{f'(z)} \right| ^2 \, d\theta \, dp \right)^{1/2} \]

where \( C \) is a bounded constant.

On the other hand, it is well known that (see for instance [3, p. 294])

\[ \int_0^{2\pi} \int_0^{2\pi} \frac{1}{\rho} \left| 1 + \frac{zf'''(z)}{f'(z)} \right| ^2 \, d\theta \, dp \leq 4\pi \log \frac{1 + r}{1 - r}. \]

This completes our proof and a question arises whether there is a positive constant \( \alpha \) and a convex function \( f(z) \) for which

\[ L(r) \geq \alpha \left( A(r) \log \frac{1}{1 - r} \right)^{1/2} \quad \text{as } r \to 1. \]

I can not give an answer for this question.

2. A function \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) that is regular in \( |z| < 1 \) is called a Bazilević function of type \( \beta \), if there exists a starlike function \( g(z) \) and \( \beta > 0 \) such that

\[ \text{Re} \frac{zf'(z)}{f(z)^{1-\beta} g(z)^{\beta}} > 0 \quad \text{in } |z| < 1. \]
Bazilevič [1] has shown that each such function is univalent in $|z| < 1$. Every starlike function is a Bazilevič function of type $\beta$.

Then the following theorems have been obtained in [6], [7] and [2], [5].

**Theorem A.** Let $f(z)$ be a Bazilevič function of type $\beta$, $0 < \beta \leq 1$ and let

$$M(r) = \max_{|z| = r} |f(re^{i\theta})| \leq (1 - r)^{-\alpha} \quad 0 < \alpha \leq 2.$$ 

Then

$$L(r) = O(1 - r)^{-\alpha} \quad \text{as } r \to 1.$$ 

**Theorem B.** Let $f(z)$ be a Bazilevič function of type $\beta$, $\arg f(z)$ be a function of bounded variation on $|z| = r < 1$ and let

$$M(r) = \max_{|z| = r} |f(re^{i\theta})| \leq (1 - r)^{-\alpha} \quad 0 < \alpha \leq 2.$$ 

Then we have

$$L(r) = O(1 - r)^{-\alpha} \quad \text{as } r \to 1.$$ 

**Theorem 2.** The results of Theorem A and Theorem B are sharp for each $\alpha$, $0 < \alpha \leq 2$ and therefore $O$ in (2) and (3) cannot be replaced by $o$.

**Proof.** It is easily verified that the function

$$f(z) = \frac{z}{(1 - z)^{\alpha}}, \quad 0 < \alpha \leq 2$$

is a starlike function [2, p. 216] and

$$M(r) \leq (1 - r)^{-\alpha}.$$ 

Applying the theorem of Fejér and Riesz to $f(z) = z/(1 - z)^{\alpha}$ we have

$$L(r) = \int_{0}^{2\pi} |zf'(z)| \, d\theta = \int_{0}^{2\pi} \left| z \left| \frac{1 - z + \alpha z}{(1 - z)^{\alpha + 1}} \right| \right| \, d\theta$$

$$\leq 2\pi \int_{-r}^{r} \frac{1 - \rho + \alpha \rho}{(1 - \rho)^{\alpha + 1}} \, d\rho$$

$$= O(1 - r)^{-\alpha} \quad \text{as } r \to 1.$$ 

This completes our proof.

The author would like to acknowledge helpful comments made by the referee.
REFERENCES


Gunma University, Japan