A NOTE ON CONVEX AND BAZILEVIČ FUNCTIONS

M. NUNOKAWA

1. Let $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ be regular, univalent in $|z|<1$ and map $|z|<1$ onto a domain which is starlike with respect to the origin. Then we call $f(z)$ a starlike function. It is well known that a function $f(z)$ is starlike in $|z|<1$ if and only if

$$\operatorname{Re}\frac{zf'(z)}{f(z)}>0 \qquad \text{in } |z|<1.$$

Let $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ be regular, univalent in $|z|<1$ and map $|z|<1$ onto a convex domain. Then we call $f(z)$ a convex function. It is well known that a regular function $f(z)$ is convex if and only if

$$1+\operatorname{Re}\frac{zf''(z)}{f'(z)}>0 \qquad \text{in } |z|<1.$$

Every convex function is a starlike function [4].

Let $L(r)$ denote the length of the closed curve $C(r)$ which is the image of the circle $|z|=r<1$ under the mapping $w=f(z)$ and $A(r)$ the area enclosed by $C(r)$.

Recently Thomas [6], [8] has shown that if $f(z)$ is a starlike function, then

$$L(r) \leqq 2(\pi A(r))^{1/2}\left(1+\log\frac{1+r}{1-r}\right). \tag{1}$$

In this note, for the convex functions we obtain a stronger result than (1).

THEOREM 1. *Let $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ be a convex function. Then we have*

$$L(r)=0\left(A(r)\log\frac{1}{1-r}\right)^{1/2} \qquad \text{as } r\to 1.$$

PROOF. Since $f(z)$ is a univalent function we can get

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$$L(r) = \int_0^{2\pi} |zf'(z)|\, d\theta \leqq \int_0^{2\pi}\int_0^r |zf''(z) + f'(z)|\, d\rho d\theta$$
$$= \int_0^r\int_0^{2\pi} \left| f'(z)\left(1 + \frac{zf''(z)}{f'(z)}\right)\right| d\theta d\rho$$
$$= \int_0^{r_1}\int_0^{2\pi} \left| f'(z)\left(1 + \frac{zf''(z)}{f'(z)}\right)\right| d\theta d\rho$$
$$+ \int_{r_1}^r\int_0^{2\pi} \left| f'(z)\left(1 + \frac{zf''(z)}{f'(z)}\right)\right| d\theta d\rho$$

where r_1 is a fixed constant and $0 < r_1 < r < 1$.

Therefore we have

$$L(r)$$
$$\leqq C + \left(\int_{r_1}^r\int_0^{2\pi} \rho\, |f'(z)|^2 d\theta d\rho\right)^{1/2} \left(\int_{r_1}^r\int_0^{2\pi} \frac{1}{\rho}\left|1 + \frac{zf''(z)}{f'(z)}\right|^2 d\theta d\rho\right)^{1/2}$$
$$\leqq C + \left(\int_0^r\int_0^{2\pi} \rho\, |f'(z)|^2 d\theta d\rho\right)^{1/2} \left(\frac{1}{r_1}\int_0^r\int_0^{2\pi} \left|1 + \frac{zf''(z)}{f'(z)}\right|^2 d\theta d\rho\right)^{1/2}$$
$$\leqq C + \left(\frac{A(r)}{r_1}\right)^{1/2} \left(\int_0^r\int_0^{2\pi} \left|1 + \frac{zf''(z)}{f'(z)}\right|^2 d\theta d\rho\right)^{1/2}$$

where C is a bounded constant.

On the other hand, it is well known that (see for instance [**3**, p. 294])

$$\int_0^r\int_0^{2\pi} \left|1 + \frac{zf''(z)}{f'(z)}\right|^2 d\theta d\rho \leqq 4\pi \log \frac{1+r}{1-r}\,.$$

This completes our proof and a question arises whether there is a positive constant α and a convex function $f(z)$ for which

$$L(r) \geqq \alpha\left(A(r) \log \frac{1}{1-r}\right)^{1/2} \qquad \text{as } r \to 1.$$

I can not give an answer for this question.

2. A function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ that is regular in $|z| < 1$ is called a Bazilevič function of type β, if there exists a starlike function $g(z)$ and $\beta > 0$ such that

$$\operatorname{Re} \frac{zf'(z)}{f(z)^{1-\beta} g(z)^{\beta}} > 0 \qquad \text{in } |z| < 1.$$

Bazilevič [1] has shown that each such function is univalent in $|z|<1$. Every starlike function is a Bazilevič function of type β.

Then the following theorems have been obtained in [6], [7] and [2], [5].

THEOREM A. *Let $f(z)$ be a Bazilevič function of type β, $0<\beta\leqq 1$ and let*

$$M(r) = \max_{|z|=r} |f(re^{i\theta})| \leqq (1-r)^{-\alpha} \qquad 0<\alpha\leqq 2.$$

Then

$$L(r) = O(1-r)^{-\alpha} \qquad \text{as } r\to 1. \tag{2}$$

THEOREM B. *Let $f(z)$ be a Bazilevič function of type β, $\arg f(z)$ be a function of bounded variation on $|z|=r<1$ and let*

$$M(r) = \max_{|z|=r} |f(re^{i\theta})| \leqq (1-r)^{-\alpha} \qquad 0<\alpha\leqq 2.$$

Then we have

$$L(r) = O(1-r)^{-\alpha} \qquad \text{as } r\to 1. \tag{3}$$

THEOREM 2. *The results of Theorem* A *and Theorem* B *are sharp for each α, $0<\alpha\leqq 2$ and therefore O in* (2) *and* (3) *can not be replaced by o.*

PROOF. It is easily verified that the function

$$f(z) = \frac{z}{(1-z)^{\alpha}}, \qquad 0<\alpha\leqq 2$$

is a starlike function [2, p. 216] and

$$M(r) \leqq (1-r)^{-\alpha}.$$

Applying the theorem of Fejér and Riesz to $f(z)=z/(1-z)^{\alpha}$ we have

$$\begin{aligned} L(r) &= \int_0^{2\pi} |zf'(z)|\,d\theta = \int_0^{2\pi} |z| \left|\frac{1-z+\alpha z}{(1-z)^{\alpha+1}}\right| d\theta \\ &\geqq 2r\int_{-r}^{r} \frac{1-\rho+\alpha\rho}{(1-\rho)^{\alpha+1}}\,d\rho \\ &= O(1-r)^{-\alpha} \qquad \text{as } r\to 1. \end{aligned}$$

This completes our proof.

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Gunma University, Japan