ON DISCONJUGACY CRITERIA

PHILIP HARTMAN[1]

1. Disconjugacy criteria. In the differential equation,

$$x^{(n)} + p_0(t)x^{(n-1)} + \cdots + p_{n-2}(t)x' + p_{n-1}(t)x = 0, \tag{1.1}$$

let the coefficients be real-valued and summable on an interval I. The equation (1.1) is said to be disconjugate on I if no solution $x(t)\not\equiv 0$ has n zeros, counting multiplicities, on I. The object of this note is to derive disconjugacy criteria, for a compact interval $I = [a, b]$ of length $|I| = b-a$, generally related to conditions of the type

$$\sum_{k=0}^{n-1} A_k |I|^k \|p_k\| \leqq 1, \tag{1.2}$$

where $A_0, \cdots, A_{n-1}$ are constants and $\| \cdots \|$ is the $L^1(I)$ norm. For $n=2$, a known disconjugacy condition is

$$2^{-1}\|p_0\|_1 + 2^{-2}|I| \cdot \|p_1^+\| \leqq 1, \tag{1.3}$$

where $r^+ = \max(0, r)$ and $r^- = \max(0, -r)$, and follows from Levin's condition [5],

$$(\exp \tfrac{1}{2}\|p_0\|_1)\|p_1^+\| \leqq 4/|I|, \tag{1.4}$$

by virtue of $e^{-x} \geqq 1-x$ for $x \geqq 0$; cf. also [0]. If $p_0 \equiv 0$, then (1.3) and (1.4) reduce to a condition of Lyapunov; e.g., [1, p. 346].

Introduce the constants

$$C_n = C(n-1, [n/2])/n!, \tag{1.5}$$

where $[r]$ is the largest integer not exceeding r and $C(n,j) = n!/j!(n-j)!$ is the binomial coefficient. These numbers $C_1 = 1$, $C_2 = 1/2$, $C_3 = 1/3$, $C_4 = 1/8, \cdots$ appear in Levin [3], [4] and Hukuhara [2]. Hukuhara [2] gives a disconjugacy criterion related to (1.2) with

$$A_0 = 1 \quad \text{and} \quad A_k = 2^{-k}C_k \qquad \text{for } k = 1, \cdots, n-1. \tag{1.6}$$

Using his method, this will be sharpened to (1.12) below.

In [6], Nehari states that (1.2) is a disconjugacy criterion if

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$$A_k = 2^{-k-1} \qquad \text{for } k = 0, \cdots, n-1. \tag{1.7}$$

It has been noted by Nehari (see [0]) that his proof of this assertion is incorrect and that the validity of the assertion is undecided. It will be seen that Nehari's statement is correct and, in fact, is contained in (1.9) below. For, on the one hand, a simple calculation shows that $(C_k/4)[k/(k+1)]^{k-1} \leqq 2^{-k-1}$ for $k=1, 2, 3$ and, on the other hand, it is easy to verify that $C_k/4 < 2^{-k-1}$ if $k \geqq 4$. (In [7], Zaĭceva asserts a refinement of criterion (1.7), but her inequality (16) on p. 765 does not seem correct, and this invalidates her proof.)

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THEOREM 1. *The following are disconjugacy criteria for* (1.1) *on* $I = [a, b]$:

(i) $p_0, \cdots, p_{n-1}$ *satisfy*

$$(\exp \tfrac{1}{2} \|p_0\|) \sum_{k=1}^{n-1} (C_k/4)[k/(k+1)]^{k-1} |I|^k \|p_k\| \leqq 1, \tag{1.8}$$

in particular,

$$\tfrac{1}{2} \|p_0\| + \sum_{k=1}^{n-1} (C_k/4)[k/(k+1)]^{k-1} |I|^k \|p_k\| \leqq 1; \tag{1.9}$$

(ii) *if, in addition,* $p_1(t) \equiv \cdots \equiv p_{m-1}(t) \equiv 0$ *for some* m, $2 \leqq m \leqq n-1$, *then* (1.8) *can be relaxed to*

$$\exp(\tfrac{1}{2} \|p_0\|) \sum_{k=m}^{n-1} (C_k/2^{m+1})[k/(k+m)]^{k-m} |I|^k \|p_k\| \leqq 1; \tag{1.10}$$

(iii) $p_0, \cdots, p_{n-1}$ *satisfy*

$$\begin{aligned} &(\exp \|p_0\|) \left\{ \sum_{k=1}^{n-1} [(C_k/2^{k+1}) |I|^k \|p_k\|]^2 \right. \\ &\qquad \left. + \tfrac{1}{2} \sum_{j \neq k} \sum C_j C_k j^j k^k (j+k)^{-j-k} |I|^{j+k} \|p_j\| \cdot \|p_k\| \right\} \leqq 1; \end{aligned} \tag{1.11}$$

(iv) *the quantities* Q_1, Q_2 *satisfy*

$$\max(Q_1, Q_2) \leqq 1, \tag{1.12}$$

where

$$Q_1 = \left(\exp \int_a^{(a+b)/2} p_0^+ dt\right) \sum_{k=1}^{n-1} C_k 2^{-k} |I|^k \int_a^{(a+b)/2} |p_k| \, dt, \tag{1.13}$$

$$(1.14)\qquad Q_2 = \left(\exp\int_{(a+b)/2}^{b} p_0^- dt\right)\sum_{k=1}^{n-1} C_k 2^{-k}\,|\,I\,|^k \int_{(a+b)/2}^{b} |\,p_k\,|\,dt.$$

2. **The Levin inequalities.** Theorem 1 will be deduced from the following inequalities given by Levin [4] and Hukuhara [3]. (The latter also makes the assumption $a \leqq a_0 \leqq a_1$.)

LEMMA 2.1_n. *Let* $n \geqq 1$, $I = [a, b]$, *and* $x(t) \in C^n(I)$. *Let* $x(t)$, $x'(t), \cdots, x^{(n-1)}(t)$ *vanish at points of* I, *say*

$$(2.1)\qquad x^{(k)}(a_k) = 0 \qquad \text{for } k = 0, \cdots, n-1,$$

and let

$$(2.2)\qquad a \leqq a_1 \leqq \cdots \leqq a_{n-1} \leqq b.$$

Then

$$(2.3)\qquad \max_I |\,x(t)\,| \leqq MC_n\,|\,I\,|^n, \quad \text{where} \quad M = \max_I |\,x^{(n)}\,|$$

and C_n *is defined by* (1.5). C_n *cannot be replaced by a smaller constant.*

Levin's proof [4] uses some functional analysis and the Krein-Milman theorem on extreme points of convex sets. Hukuhara's proof [2] is an elementary calculus proof. We shall give a different, short, elementary proof which will yield sharper inequalities (cf. (2.6)–(2.7), (2.8)–(2.9), (3.5) and (4.3) below) and has the following consequence:

COROLLARY 2.1. *The assertion* (2.3) *in Lemma* 2.1_n *can be strengthened to*

$$(2.4)\qquad \int_a^b |\,x'(t)\,|\,dt \leqq MC_n\,|\,I\,|^n,$$

whether or not $x(t)$ *has a zero* $t = a_0$ *on* I.

REMARK. It is easy to verify that equality in (2.3) holds if $|x^{(n)}(t)| \equiv M$, $a_k = a_{n-1}$ for $k \geqq n/2$, $a_k = a_0$ for $k < n/2$, and $a = a_0$, $b = a_{n-1}$. Furthermore, the proof of Lemma 2.2 will imply that equality cannot hold in any other cases; cf. $(2.6_n)-(2.7_{nk})$ and (3.5) below.

LEMMA 2.2_n. *Strengthen* (2.2) *in Lemma* 2.1_n *to*

$$(2.5)\qquad a \leqq a_0 \leqq \cdots \leqq a_{n-1} \leqq b \quad \text{and let } a_n = b.$$

Then, on $[a, a_0]$,

$$(2.6_n)\qquad |\,x(t)\,| \leqq \int_t^b |\,x'(s)\,|\,ds \leqq \int_t^b |\,x^{(n)}(s)\,|\,(s-t)^{n-1}ds/(n-1)\,!;$$

and, on $[a_k, a_{k+1}]$, *for* $k=0, \cdots, n-1$,

$$(2.7_{nk})\qquad \begin{aligned} (n-2)!\,|x(t)| &\leqq (n-2)!\int_{a_0}^{t} |x'(s)|\,ds \\ &\leqq \sum_{j=0}^{k-1} C(n-2, j)\int_{a_j}^{a_{j+1}} (t-r)^j F_j(r)dr \\ &\quad + C(n-2, k)\int_{a_k}^{t} (t-r)^k F_k(r)dr, \end{aligned}$$

where

$$F_j(r) = \int_r^b |x^{(n)}(s)|\,(s-r)^{n-2-j}ds.$$

In particular, if $M = \max|x^{(n)}|$ *on* $[a, b]$, *then*

$$(2.8)\qquad n!\,|x(t)| \leqq M[(b-t)^n - (b-a_0)^n] \qquad on\ [a, a_0],$$

$$(2.9)\qquad n!\,|x(t)| \leqq M[(b-a_0)^n - (b-t)^n] \max_{0\leqq j\leqq k} C(n-1, j)$$

on $[a_k, a_{k+1}]$ *for* $k=0, 1, \cdots, n-1$.

Although the inequalities (2.6_n), (2.7_{nk}) may not be in a very useful form, the point of these relations is that they have a simple proof and imply (2.8), (2.9) and Lemma 2.1.

3. **Proof of Lemma 2.2.** *On* (2.6_n). The relation (2.6_1) is trivial on $[a, a_0]$. Assume $n \geqq 2$ and (2.6_{n-1}). Thus (2.6_{n-1}) applied to $x'(t)$ gives

$$(3.1)\qquad |x'(t)| \leqq \int_t^b |x^{(n)}(s)|\,(s-t)^{n-2}ds/(n-2)! \qquad \text{on } [a, a_1].$$

For $a \leqq t \leqq a_0$, the inequality

$$|x(t)| \leqq \int_t^{a_0} |x'(r)|\,dr \leqq \int_t^b |x'(r)|\,dr$$

combined with (3.1) gives (2.6_n).

On (2.7_{nk}). The inequality (3.1) and $|x(t)| \leqq \int_{a_0}^t |x'(r)|\,dr$ for $t \geqq a_0$ gives (2.7_{n0}) for $n=2, 3, \cdots$. Assume (2.7_{nk}) for fixed $k \geqq 0$ and all $n \geqq k+2$. We shall verify $(2.7_{n,k+1})$. From (2.7_{nk}) for $t=a_{k+1}$,

$$(3.2)\qquad \begin{aligned} &(n-2)!\int_{a_0}^{a_{k+1}} |x'(s)|\,ds \\ &\qquad \leqq \sum_{j=0}^{k} C(n-2, j)\int_{a_j}^{a_{j+1}} (a_{k+1}-r)^j F_j(r)dr; \end{aligned}$$

and $(2.7_{n-1,k})$ applied to x' gives

$$(3.3)\qquad \begin{aligned}(n-3)!\,|x'(v)| \leqq \sum_{j=0}^{k-1} C(n-3,j)\int_{a_{j+1}}^{a_{j+2}} (v-r)^j F_{j+1}(r)dr \\ + C(n-3,k)\int_{a_{k+1}}^{v} (v-r)^k F_{k+1}(r)dr\end{aligned}$$

for $a_{k+1} \leqq v \leqq a_{k+2}$ and $k=0, \cdots, n-2$. A quadrature of (3.3),

$$(3.4)\qquad \begin{aligned}(n-2)!\int_{a_{k+1}}^{t} |x'(v)|\,dv \\ \leqq \sum_{j=1}^{k} C(n-2,j)\int_{a_j}^{a_{j+1}} [(t-r)^j - (a_{k+1}-r)^j]F_j(r)dr \\ + C(n-2,k+1)\int_{a_{k+1}}^{t} (a_{k+1}-r)^{k+1}F_{k+1}(r)dr,\end{aligned}$$

together with (3.2) gives $(2.7_{n,k+1})$.

On (2.8). By (3.1), $|x'(t)| \leqq M(b-t)^{n-1}/(n-1)!$ on $[a, a_1]$. A quadrature over $[t, a_0]$ gives (2.8).

On (2.9). The right side of (2.7_{nk}) is not decreased if $|x^{(n)}(s)|$ is replaced by M and, in the integrand, t is replaced by b. Thus, on $[a_k, a_{k+1}]$,

$$(3.5)\qquad \begin{aligned}n!\,|x(t)| \leqq M\sum_{j=0}^{k-1} C(n-1,j)[(b-a_j)^n - (b-a_{j+1})^n] \\ + MC(n-1,k)[(b-a_k)^n - (b-t)^n].\end{aligned}$$

This imples (2.9) and completes the proof of Lemma 2.2.

4. **Proof of Corollary 2.1.** The proof of (2.7_{nk}) shows that if $a \leqq a_0 \leqq a_1$ and $a_k \leqq t \leqq a_{k+1}$, then

$$(4.3)\qquad \begin{aligned}n!\int_{a_0}^{t} |x'(s)|\,ds \leqq M\sum_{j=0}^{k-1} C(n-1,j)[(b-a_j)^n - (b-a_{j+1})^n] \\ + MC(n-1,k)[(b-a_k)^n - (b-t)^n],\end{aligned}$$

whether or not $x(a_0)=0$. This inequality, with $a=a_0$, $k=n-1$, and $t=b=a_n$, implies (2.4), since $n!\,C_n = \max C(n-1,j)$ for $0 \leqq j \leqq n-1$.

5. **Proof of Theorem 1.** Suppose that (1.1) has a solution $x(t) \not\equiv 0$ with n zeros, counting multiplicities, on $[a, b]$. It will be shown that the inequalities (1.8), (1.10), (1.11) and (1.12) cannot hold.

We can suppose that there are numbers

$$a = a_0 \leqq a_1 \leqq \cdots \leqq a_{n-1} = b_{n-1} \leqq b_{n-2} \leqq \cdots \leqq b_1 \leqq b_0 = b,$$

such that $x^{(k)}(t)=0$ if $t=a_k$ and $t=b_k$. Put $c=a_{n-1}=b_{n-1}$, $a<c<b$. The analogues of Lemma 2.1_{n-1} are applicable to each of the intervals $[a, c]$, $[c, b]$. Actually, the analogues will be applied to $x^{(n-1-k)}$, rather than to x with $n-1$ replaced by k, $k=1, \cdots, n-2$. Let $M=\max|x^{(n-1)}(t)|$ on $[a, c]$ and let $|x^{(n-1)}(r)|=M$, $a<r<c$.

Write (1.1) as

$$\left(x^{(n-1)}\exp\int_a^t p_0 ds\right)' + \left(\exp\int_a^t p_0 ds\right)\sum_{k=1}^{n-1} p_k(t)x^{(n-1-k)} = 0.$$

An integration over the interval $[r, c]$ gives

$$M\exp\int_a^r p_0 ds < \sum_{k=1}^{n-1}\max_{[a,c]}|x^{(n-1-k)}|\int_r^c\left(\exp\int_a^t p_0 ds\right)|p_k|\,dt.$$

By Lemma 2.1_k and

$$\exp\int_r^t p_0 ds \leqq \exp\int_a^c p_0^+ ds \qquad \text{for } r\leqq t\leqq c,$$

we get

$$1 < \left(\exp\int_a^c p_0^+ ds\right)\sum_{k=1}^{n-1} C_k(c-a)^k\int_a^c |p_k|\,dt. \tag{5.1}$$

Similarly,

$$1 < \left(\exp\int_c^b p_0^- ds\right)\sum_{k=1}^{n-1} C_k(b-c)^k\int_c^b |p_k|\,dt. \tag{5.2}$$

On (1.12). Since either $c\leqq(a+b)/2$ or $c\geqq(a+b)/2$, that is, $c-a\leqq(b-a)/2$ or $b-c\leqq(b-a)/2$, (5.1) and (5.2) imply that either $Q_1>1$ or $Q_2>1$.

On (1.8). By (5.1) and (5.2), we have

$$\begin{aligned}1 &< (\exp\|p_0\|)\sum_{j=1}^{n-1}\sum_{k=1}^{n-1} C_jC_k(c-a)^k(b-c)^j \\ &\quad\cdot\int_a^c |p_k|\,dt\int_c^b |p_j|\,dt.\end{aligned} \tag{5.3}$$

Note that

$$(c-a)^k(b-c)^j \leqq |I|^{k+j}k^kj^j/(k+j)^{k+j} \qquad \text{for } a\leqq c\leqq b \tag{5.4}$$

If $m\geqq 0$, then $jk/(j+k)^2\leqq 1/4$ implies that

$$
\begin{aligned}
j^j k^k/(j+k)^{j+k} &= [jk/(j+k)^2]^m j^{j-m} k^{k-m}/(j+k)^{j+k-2m} \\
&\leqq 2^{-2m}[j/(j+k)]^{j-m}[k/(j+k)]^{k-m}.
\end{aligned} \tag{5.5}
$$

If $j,\ k \geqq i$ and $j,\ k \geqq m \geqq 0$, then

$$
j^j k^k/(j+k)^{j+k} \leqq 2^{-2m}[j/(j+i)]^{j-m}[k/(k+i)]^{k-m}. \tag{5.6}
$$

The inequalities (5.3), (5.4) and (5.6), with $i=m=1$, show that

$$
\begin{aligned}
1 < (\exp \|p_0\|) &\left\{ \sum_{k=1}^{n-1} (C_k/2)[k/(k+1)]^{k-1} |I|^k \int_a^c |p_k|\, dt \right\} \\
\times &\left\{ \sum_{j=1}^{n-1} (C_j/2)[j/(j+1)]^{j-1} |I|^j \int_c^b |p_j|\, dt \right\}.
\end{aligned}
$$

Take the square root of both sides of this inequality and use the arithmetic-geometric mean inequality, applied to the last two factors, to obtain

$$
1 < (\exp \tfrac{1}{2} \|p_0\|) \sum_{k=1}^{n-1} (C_k/4)[k/(k+1)]^{k-1} |I|^k \int_a^b |p_k|\, dt.
$$

On (1.10). This is proved in the same way using $i=m$ in (5.6).

On (1.11). Insert the inequality (5.4) into (5.3), interchange j and k, and add the resulting inequalities to obtain

$$
\begin{aligned}
2 < (\exp \|p_0\|) &\sum_{j=1}^{n-1} \sum_{k=1}^{n-1} C_j C_k j^j k^k (j+k)^{-j-k} |I|^{j+k} \\
&\cdot \left\{ \int_a^c |p_k|\, dt \int_c^b |p_j|\, dt + \int_c^b |p_k|\, dt \int_a^c |p_j|\, dt \right\}.
\end{aligned}
$$

The factors $\{ \cdots \}$ satisfy

$$
\{ \cdots \} \leqq \|p_j\| \cdot \|p_k\| \qquad \text{for all } j,\ k;
$$

but if $j=k$,

$$
\{ \cdots \} = 2 \int_a^c |p_k|\, dt \int_c^b |p_k|\, dt \leqq \tfrac{1}{2}(\|p_k\|)^2 \qquad \text{for } j = k.
$$

The last three formula lines contradict (1.11) and complete the proof of Theorem 1.

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