SHORTER NOTE

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AN INEQUALITY FOR RATIONAL FUNCTIONS

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Abstract. An inequality of A. A. Gončar concerning the relative sizes of rational functions on different sets is reinterpreted. This allows a very simple proof of a more general inequality.

A. A. Gončar [1] proved the following: Let $G$ be a doubly connected domain in the Riemann sphere bounded by continua $E_1$ and $E_2$. Then if $r$ is any rational function of degree (number of poles) $n$, one has

$$\min_{z \in E_2} |r(z)| \leq \rho^n \max_{z \in E_1} |r(z)|$$

where $\rho$ is the modulus of the domain $G$. (This means that $G$ is conformally equivalent to the annulus $1 < |z| < \rho$.)

In this note we present a simple proof of the generalization of this inequality to arbitrary disjoint closed sets $E_1$ and $E_2$ of positive (logarithmic) capacity.

Let $g(z, \xi)$ denote Green's function for the complement of $E_1$ and define $1/\log \rho$ to be the capacity of $E_2$ relative to this kernel. Thus $1/\log \rho$ is the maximum of $\mu(E_2)$ for all nonnegative Borel measures $\mu$ on $E_2$ satisfying

$$\int g(z, \xi) d\mu(z) \leq 1, \quad \xi \in E_1.$$  

It is a fact, not difficult to prove, that $\rho$ is unchanged if the roles of $E_1$ and $E_2$ are interchanged. In case $E_1$ and $E_2$ bound a doubly connected domain $G$ then $\rho$ as just defined is the modulus of $G$ [2, pp. 96–97].

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We shall show that (1) holds in the general case. We may assume that the maximum appearing on the right side of (1) is 1. If $r$ has poles at $\xi_1, \ldots, \xi_n$ then

$$\log |r(z)| - \sum_{k=1}^{n} g(z, \xi_k)$$

is subharmonic in the complement of $E_1$ and has lim sup at most 0 at each point of $E_1$. Therefore by the maximum principle for subharmonic functions

$$\log |r(z)| \leq \sum_{k=1}^{n} g(z, \xi_k) \quad z \in E_1.$$

Now let $\mu$ be any measure on $E_2$ satisfying (2). Then

$$\mu(E_2) \min_{z \in E_2} \log |r(z)| \leq \sum_{k=1}^{n} \int g(z, \xi_k) d\mu(z) \leq n,$$

and (1) follows immediately from the definition of $\mu$.

It can be shown [3] that the constant $\rho$ appearing on the right side of (1) is best possible.

References


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