# CANTOR SETS IN METRIC MEASURE SPACES

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The discussions and results below are the fruits of a number of conversations the author held with Harry Furstenberg, Richard Darst and Robert Zink. The burden of the paper is the phenomenon that may be paraphrased (approximately) as follows: *Every compact metric measure space is the union of the Cantor set and an open set of small measure.* The preceding sentence is *false.* Its *correct* version is the theorem below.

We preface the argument with the following remarks, many of which, despite their elementary nature, are made for completeness.

a. A compact metric space $X$ is separable. It is either countable or its cardinality card $(X)$ is $\mathfrak{c}\equiv$ card $(\boldsymbol{R})$. If card $(X)=\mathfrak{c}$, then $X$ contains a homeomorphic image of the Cantor set [1].

b. If $S$ is the $\sigma$-ring generated by the compact sets of the compact metric space $X$, and if $\mu$ is a finite nonatomic measure defined for the elements of $S$, then $\mu$ is regular [2]. In what follows, we assume that $X$ is a compact metric space, that $\mu$ is a nontrivial, finite nonatomic measure defined for the elements of $S$. Thus card$(X)=\mathfrak{c}$ and $\mu$ is regular. The next result was suggested by Darst. The notations above prevail.

LEMMA 1. *There is a countable neighborhood basis $\mathfrak{U}=\{U_n\}$ for the topology of $X$ and such that $\mu(\partial U_n)=0$ for all $U_n\in\mathfrak{U}$, where $\partial U_n$ denotes the boundary of $U_n$.*

PROOF. Let $\rho$ be the metric for $X$ and let $S_{x,d}=\{y:\rho(x,y)<d\}$, for $d>0$, $x\in X$. For each $x$, the boundaries $\partial_{x,d}\equiv\partial S_{x,d}$ are disjoint for distinct $d$: $d_1\neq d_2$ implies $\partial_{x,d_1}\cap\partial_{x,d_2}=\emptyset$. Since $\mu$ is finite, for only countably many $d$ is $\mu(\partial_{x,d})>0$. For each $x$ let $S_{x,d_n(x)}$ be such that $0<d_{n+1}(x)<d_n(x)$, $d_n(x)\to 0$ as $n\to\infty$ and such that $\mu(\partial_{x,d_n(x)})=0$. For each $m=1,2,\cdots$, let $d_{n_m}(x)$ be the first $d_n(x)$ under $1/m$: $d_{n_m}(x)=\sup\{d_n(x):d_n(x)<1/m\}$. Then for each $m$ there is a finite cover $\{S_{x_{mj},\,d_{n_m}(x_{mj})}:j=1,2,\cdots,J_m\}$ of $X$. Let $\mathfrak{U}$ be the set of all

$$S_{x_{mj},\,d_{n_m}(x_{mj})},\qquad m=1,2,\cdots,j=1,2,\cdots,J_m.$$

Then $\mathfrak{U}$ is a countable family of open sets, the sets of $\mathfrak{U}$ constitute a cover for $X$ and, as we now show, $\mathfrak{U}$ is a basis for the topology of $X$.

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Indeed, if $x \in X$ and if $V$ is an open set containing $x$, let $S_{x,d} \subset V$. Then let $2/m < d$ and choose $S \equiv S_{x_{mj}, d_{n_m}(x_{mj})}$ containing $x$. Then since the diameter of $S$ does not exceed $2/m$ we see that $x \in S \subset S_{x,d} \subset V$. Thus $\mathfrak{U}$ is a countable basis for the topology of $X$.

LEMMA 2. *If $Y$ is a Hausdorff space, if $\mathfrak{U} = \{U\}$ is a neighborhood basis for $Y$ and if $V_U$ is an open set containing $\partial U$ for each $U$, then $Y \setminus \bigcup_{\mathfrak{U}} V_U$ is either empty, a one-point set or totally disconnected.*

PROOF. If $Y \setminus \bigcup_{\mathfrak{U}} V_U$ contains more than one point, let $Z$ be a connected subset of $Y \setminus \bigcup_{\mathfrak{U}} V_U$, where card $(Z) > 1$. Then since $Y \setminus \bigcup_{\mathfrak{U}} V_U$ is closed, we see that we may assume $Z$ is closed. Choose a point $z$ in $Z$ and a neighborhood $U \in \mathfrak{U}$ such that $Z \setminus U \neq \emptyset$. Then $\emptyset \subset \{z\} \subset Z \cap (U \setminus V_U) = Z \cap U$ since $Z \subset Y$, and thus $Z = (Z \setminus U) \cup (Z \cap U)$, the union of two disjoint nonempty sets. Of course, $Z \setminus U$ is closed. However, $\overline{U} \setminus V_U = U \setminus V_U$ since if $x \in \overline{U} \setminus V_U$ and if $x \notin U$, then $x \in \partial U$, whence $x \in V_U$ whence $x \notin \overline{U} \setminus V_U$. Thus $Z \cap U = Z \cap (\overline{U} \setminus V_U)$ is closed and so $Z$ cannot be connected.

THEOREM. *Let $X$ be a compact metric space; let $\mu$ be a nontrivial, nonatomic, finite measure defined on the Borel sets of $X$. For each $\epsilon > 0$ there is an open set $W$ such that $(X \setminus W)$ is homeomorphic to the Cantor set and $\mu(W) < \epsilon$.*

PROOF. Let $\{U_n\}$ be a countable basis that is a subset of $\mathfrak{U}$. Since $\mu$ is regular we may choose an open set $V_n \supset \partial U_n$ and such that $\mu(V_n) < \epsilon/2^n$, $n = 1, 2, \cdots$. Then $X \setminus \bigcup_n V_n$ is totally disconnected, compact and hence, by the Cantor-Bendixson theorem [**3**], the union of a perfect set $P$ and a countable set $D$. We may assume $\mu(X \setminus \bigcup_n V_n) > 0$ whence $P$ is nonempty and thus $P$ and $C$ are homeomorphic [**1**]. The open set $W = X \setminus P$ has measure less than $\sum_n \mu(V_n) + \mu(D) < \epsilon$.

*Note.* Owing to the prefatory remarks, we can conclude that if card $(X) = \mathfrak{c}$, then $X$ contains a homeomorphic image $P$ of the Cantor set on which there lives a finite measure $\mu$. In this instance $\mu(X \setminus P) = 0$. The impact of the theorem is that if the measure $\mu$ is given *a priori*, then for some $P$, $\mu(X \setminus P) < \epsilon$.

In a forthcoming paper [**4**], Oxtoby, dealing with related topics, shows that the theorem above conjoined with a result in [**2**] yields:

THEOREM. *Let $X$ be a complete separable metric space, let $\mu$ be a nonatomic Borel measure in $X$, and let $B$ be a Borel set with $0 < \mu(B) < \infty$. For each $\epsilon > 0$ there is a set $C \subset B$ such that $C$ is homeomorphic to the Cantor set and $\mu(B \setminus C) < \epsilon$.*

PROOF. By [2, p. 183, (3e)] there exists a compact set $K \subset B$ such that $\mu(K) > \mu(B) - \epsilon$. By the above theorem, $K$ contains a set $C$ homeomorphic to the Cantor set with $\mu(C) > \mu(B) - \epsilon$.

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