Q-UNIFORM BANACH ALGEBRAS[1]

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0. Introduction. The Gelfand-Mazur theorem for commutative Banach algebras may be stated as follows:

If A is a commutative Banach algebra and if M_1 and M_2 are two regular maximal ideals of A then A/M_1 and A/M_2 are $\mathbf{C}$-isomorphic.

A class of not necessarily commutative Banach algebras generalizing commutative Banach algebras is motivated by the above result and is defined as follows:

A Banach algebra A is called a Q-uniform Banach algebra if:

(i) Q is a simple Banach algebra with identity e_Q.

(ii) A is a Q-bimodule. Thus for a, $b\in A$, q, $q'\in Q$ qa and aq are (not necessarily) equal elements of A; $(q, a)\to qa$, $(q, a)\to aq$ are bilinear;

$$e_Q a = ae_Q = a;\ q(ab) = (qa)b,\ (ab)q = a(bq),\ (qa)q' = q(aq'),\ q(q'a) = (qq')a,\ a(qq') = (aq)q';\ \|qa\|, \|aq\| \leqq \|a\|\,\|q\|.$$

(iii) If M is a regular maximal ideal of A then A/M is $\mathbf{C}$-isomorphic to Q. (Alternatively, if $\eta: A\to Q_1$ is an $\mathbf{C}$-epimorphism[3] of A onto a simple Banach algebra Q_1 with identity, then Q_1 is $\mathbf{C}$-isomorphic to Q.)

A class of Q-uniform algebras may be described as follows. Let Q be an arbitrary simple Banach algebra with identity e_Q, e.g., $Q = \mathrm{End}_{\mathbf{C}}(\mathbf{C}^n) \equiv$ the set of $\mathbf{C}$-endomorphisms of $\mathbf{C}^n$. Let X be a compact Hausdorff space and let $A = C(X, Q) \equiv$ the set of Q-valued continuous functions on X. Clearly A is a Q-bimodule as defined in (ii) above. We verify (iii) in the next paragraph.

Let $M_A \in \mathfrak{M}_A \equiv$ the set of regular maximal ideals of A. We show that there is an $x_0\in X$ such that $M_A = \{f: f(x_0) = 0\}$. Otherwise there is for each x an $f_x \in M_A$ such that $q_x = f_x(x) \neq 0$. The ideal generated in Q by q_x must be Q since Q is simple and so if $\sum_{i=1}^m q_i q_x q_i' = e_Q$ then

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[3] All morphisms are assumed to be continuous. We work in the category of Banach algebras and continuous homomorphisms.

$\tilde{f}=\sum_{i=1}^{m} q_i' f_x q_i$ is such that $\tilde{f}(x)=e_Q$. Via compactness of X we conclude there are open sets $\{U_i\}_1^n$, $\bigcup U_i=X$ and functions $\{f_i\}\subset M_A$ (cf. Lemma 1.1) such that for $x\in U_i$, $\|f_i(x)-e_Q\|_Q<\frac{1}{2}$ whence $(f_i(x))^{-1}$ exists for $x\in U_i$. If $\{\phi_i\}_1^n$ is a **C**-valued partition of unity subordinate to $\{U_i\}_1^n$ then $\phi_i e_Q\in A$, $\phi_i e_Q f_i=\phi_i f_i\in M_A$ and $\sum_{i=1}^n \phi_i f_i\in M_A$. Then for any x

$$\left\| \sum_{i=1}^n \phi_i(x) f_i(x) - e_Q \right\|_Q = \left\| \sum_{i=1}^n \phi_i(x)(f_i(x) - e_Q) \right\|_Q$$
$$\leqq \sum_{i=1}^n |\phi_i(x)| \, \|f_i(x) - e_Q\|_Q.$$

However

$$\begin{aligned} |\phi_i(x)| \, \|f_i(x) - e_Q\|_Q < \tfrac{1}{2} |\phi_i(x)| &= \tfrac{1}{2}\phi_i(x), \quad \text{if } x \in U_i \\ &= 0 \leqq \tfrac{1}{2}\phi_i(x), \quad \text{if } x \notin U_i. \end{aligned}$$

Thus

$$\left\| \sum_{i=1}^n \phi_i(x) f_i(x) - e_Q \right\|_Q \leqq \frac{1}{2} \sum_{i=1}^n \phi_i(x) = \frac{1}{2} < 1$$

and we conclude

$$\left(\sum_{i=1}^n \phi_i f_i \right)^{-1}$$

exists, a contradiction. Hence for some x_0, $M_A=\{f: f(x_0)=0\}$. The map $\eta: f\to f(x_0)$ is a **C**-epimorphism of A onto Q and $\ker(\eta)=M_A$, whence A/M_A is **C**-isomorphic to Q.

1. **Fundamentals.** In this section we gather some elementary facts about Q-uniform algebras.

Lemma 1.1. *If an algebra A is a bimodule over an algebra B then every regular ideal I of A is a B-ideal.*

Proof. Let $ua-a\in I$ for all $a\in A$. Then if $b\in B$, $x\in I$ we see $u(bx)-bx\in I$, $(ub)\cdot x\in I$ and thus $bx\in I$. Similarly $xb\in I$.

Lemma 1.2. *Let A be Q-uniform and let $M_A\in\mathfrak{M}_A$, $u/M_A=e_Q$ $\equiv$ identity of $Q\cong A/M_A$.*

(i) *If every **C**-monoendomorphism α of Q such that $\alpha(e_Q)=e_Q$ is a **C**-automorphism, then $A=Qu\oplus M_A$.*

(ii) *If $A=Qu\oplus M_A$, if $\eta\in\mathrm{Epi}_{\mathbf{C}}(A, Q)\equiv$ the set of **C**-epimorphisms of A onto Q, if $\ker(\eta)=M_A$ and if $\alpha_\eta(q)=\eta(qu)$ then $\alpha_\eta\in\mathrm{Aut}_{\mathbf{C}}(Q)$ $\equiv$ the set of **C**-automorphisms of Q.*

REMARK. Every noncommutative Banach algebra B with identity e has a nontrivial group $\mathrm{Aut}_C(B)$. Indeed let $x \in B \setminus [\text{center of } B]$ and let $\epsilon > 0$ be such that $(e+\epsilon x)^{-1}$ exists. Then the inner automorphism

$$y \to (e + \epsilon x)^{-1} y (e + \epsilon x)$$

is nontrivial.

PROOF. (i) Let $\eta \in \mathrm{Epi}_C(A, Q)$ and let $\ker(\eta) = M_A$. Define $\alpha_\eta \in \mathrm{End}_C(Q)$ by $\alpha_\eta(q) = \eta(qu)$. Then α_η is clearly linear and $\mathbf{C}$-homogeneous. Furthermore, $qu - uqu$, $uq - uqu \in M_A$, whence $qu - uq \in M_A$. Thus $\alpha_\eta(q_1q_2) = \eta(q_1q_2u) = \eta(q_1q_2u^2) = \eta(q_1uq_2u) = \alpha_\eta(q_1)\alpha_\eta(q_2)$. Next $\alpha_\eta(q) = 0$ implies $qu \in M_A$. Thus $\sum_{i=1}^n q_i qu q_i' \in M_A$ for all q_i, q_i'. However, $uq_i' = q_i'u + m_i$, $m_i \in M_A$, and so $\sum_{i=1}^n q_i q q_i' u \in M_A$. Since $\{\sum_{i=1}^n q_i q q_i'\}$, if $q \neq 0$, is a nontrivial ideal in Q, it is Q and so $Qu \subset M_A$. Thus $e_Q u = u \in M_A$, a contradiction. Hence $\alpha_\eta(q) = 0$ implies $q = 0$ and α_η is injective. Finally, $\alpha_\eta(e_Q) = \eta(e_Q u) = \eta(u) = e_Q$. By hypothesis, then, $\alpha_\eta \in \mathrm{Aut}_C(Q)$.

Now for $a \in A$ consider $a - \alpha_\eta^{-1}\eta(a)u$. Then

$$\eta(a - \alpha_\eta^{-1}\eta(a)u) = \eta(a) - \alpha_\eta(\alpha_\eta^{-1}\eta(a)) = \eta(a) - \eta(a) = 0.$$

Thus $a - \alpha_\eta^{-1}\eta(a)u = m \in M_A$. We saw in the preceding paragraph that $qu \in M_A$ implies $q = 0$, i.e., $Qu \cap M_A = \{0\}$. Thus $A = Qu \oplus M_A$.

(ii) If $A = Qu \oplus M_A$ and $\ker(\eta) = M_A$ then $\eta(Qu) = \eta(A) = Q$, whence $\alpha_\eta \in \mathrm{Aut}_C(Q)$.

REMARKS. 1. The hypothesis of Lemma 1.2 is neither too restrictive nor superfluous. Clearly, if Q is finite-dimensional, the hypothesis is satisfied. As G. K. Kalisch has noted, on the other hand, let $\mathfrak{H}$ be a separable Hilbert space and let $\mathfrak{B}(\mathfrak{H})$ be the set of bounded $\mathbf{C}$-endomorphisms of $\mathfrak{H}$. If $\{\phi_n\}$ is a complete orthonormal set in $\mathfrak{H}$ the elements of $\mathfrak{B}(\mathfrak{H})$ may be represented by countably infinite matrices. Define $\beta: \mathfrak{B}(\mathfrak{H}) \to \mathfrak{B}(\mathfrak{H})$ by the rule:

$$\beta: \begin{pmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \vdots & \vdots & \end{pmatrix} \to \begin{pmatrix} a_{11} & 0 & a_{12} & 0 & \cdots \\ 0 & a_{11} & 0 & a_{12} & \cdots \\ a_{21} & 0 & a_{22} & 0 & \cdots \\ 0 & a_{21} & 0 & a_{22} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

Then β is a monoendomorphism of $\mathfrak{B}(\mathfrak{H})$, $\beta(I) = I$ and β is not epic.

Next we show that if T is compact then so is $\beta(T)$. Indeed, let $x_k = \{x_{kn}\}$ be a sequence converging weakly in $\mathfrak{H}$ to 0: $x_k \to^w 0$. Thus we may assume $|x_k|^2 \equiv \sum_{n=1}^\infty |x_{kn}|^2 \leq 1$, $x_{kn} \to 0$ as $k \to \infty$ for each n.

Thus if $T=(t_{ij})$ and $\beta(T)=(s_{ij})$ we see that

$$\beta(T)(x_k) = \left\{ \sum_j a_{1j}x_{k,2j-1}, \sum_j a_{2j}x_{k,2j}, \cdots, \sum_j a_{2p-1,j}x_{k,2j-1}, \sum_j a_{2p,j}x_{k,2j}, \cdots \right\}.$$

If we set $x_k' = \{x_{k1}, x_{k3}, \cdots\}$, $x_k''=\{x_{k2}, x_{k4}, \cdots\}$ then

$$\beta(T)(x_k) = Tx_k' + Tx_k''.$$

Clearly $x_k', x_k'' \xrightarrow{w} 0$ whence $|Tx_k'|, |Tx_k''| \to 0$, whence $|\beta(T)(x_k)| \to 0$ and so $\beta(T)$ is compact. Furthermore

$$|\beta(T)x|^2 \leqq |T|^2(|x'|^2 + |x''|^2) = |T|^2|x|^2$$

i.e., $|\beta(T)| \leqq |T|$, and so β is continuous. Clearly $\beta(I)=I$.

Thus we see that if $\mathfrak{K}$ denotes the (unique) maximal ideal of compact operators in $\mathfrak{B}(\mathfrak{H})$, then $\beta(\mathfrak{K})\subset\mathfrak{K}$ whence β may be viewed as a $\mathbf{C}$-endomorphism B of the simple quotient algebra $Q\equiv\mathfrak{B}(\mathfrak{H})/\mathfrak{K}$.

We show B is a continuous monoendomorphism of Q, that $\mathrm{B}(e_Q) = e_Q = I/\mathfrak{K}$, and that $\mathrm{B}(Q) \subsetneqq Q$. Indeed, if $q_n\in Q$, $|q_n|\to 0$, let $T_n/\mathfrak{K} = q_n$, $|T_n|\to 0$ (by virtue of the open mapping theorem). Then $\mathrm{B}(q_n)=\beta(T_n)/\mathfrak{K}\to 0$ and so B is continuous. Since $\beta(I)=I$ we see $\mathrm{B}(e_Q)=e_Q$.

If $\mathrm{B}(q)=0$, let $q=T/\mathfrak{K}$. Then $\beta(T)\in\mathfrak{K}$. In the notations used earlier, let $y_k=\{x_{k1}, x_{k1}, x_{k2}, x_{k2}, x_{k3}, x_{k3}, \cdots\}$. Then $y_k \xrightarrow{w} 0$ whence $|\beta(T)(y_k)|\to 0$. However, $|\beta(T)(y_k)|^2=2|Tx_k|^2$. Thus $T\in\mathfrak{K}$ and thus $q=T/\mathfrak{K}=0$. We conclude that B is monic.

Finally we show B is not epic: $\mathrm{B}(Q) \subsetneqq Q$. Indeed, if:

$$T \sim \begin{pmatrix} 1 & 1 & 0 & 0 & 0\cdots \\ 1 & 1 & 1 & 0 & 0\cdots \\ 0 & 1 & 1 & 1 & 0\cdots \\ 0 & 0 & 1 & 1 & 1\cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

Then $|T\{0, 0, \cdots, 1, 0, \cdots\}|^2 \geqq 2$ and we see T is not compact, $T\notin\mathfrak{K}$. Clearly $T\notin\beta(\mathfrak{B}(\mathfrak{H}))$. We show $T/\mathfrak{K}\notin\mathrm{B}(Q)$. Indeed, if $\mathrm{B}(q) = T/\mathfrak{K}$, let $q=S/\mathfrak{K}$. Then $\mathrm{B}(q)=\beta(S)/\mathfrak{K}$ whence $\beta(S)-T\in\mathfrak{K}$. If $S\sim(s_{ij})$ then:

$$\beta(S)-T\sim\begin{vmatrix} s_{11}-1, & -1, & s_{12}\quad, & 0, & s_{13}, \cdots \\ -1, & s_{11}-1, & -1, & s_{12}\quad, & 0, \cdots \\ s_{21}\quad, & -1, & s_{22}-1, & -1, & s_{23}, \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

However, $|(\beta(S)-T)\{0, 0, \cdots, 0, 1, 0, \cdots\}|^2 \geqq 1$ whence $\beta(S)-T\notin\mathfrak{K}$. The contradiction implies B is not epic. In conclusion we see that $\mathfrak{B}(\mathfrak{H})/\mathfrak{K}=Q$ is a simple Banach algebra for which there is a monoendomorphism B carrying the identity itself and yet failing to be an automorphism.

2. If $\tilde{u}/M_A=e_Q$, let $\tilde{\alpha}_\eta(q)=\eta(q\tilde{u})$. Then $\tilde{u}=u+m$, $m\in M_A$ and $q\tilde{u}=qu+qm$. However (Lemma 1.1) $qm\in M_A$, whence $\eta(q\tilde{u})=\eta(qu)$ and thus $\tilde{\alpha}_\eta=\alpha_\eta$. In other words the definition of α_η is independent of the choice of a member of $\eta^{-1}(e_Q)$.

LEMMA 1.3. *Let A be Q-uniform but without an identity. Define $A_e=\{qe+a: q\in Q,\ a\in A\}$ where e is an adjoined identity defined to satisfy $ea=ae=a$, $qe=eq$. Then A_e is Q-uniform and there is a one-one correspondence between $\mathfrak{I}_{A_e}=\{I_e\}$, the set of ideals of A_e and not contained in A and $\mathfrak{I}_A=\{I\}$, the set of regular ideals of A. In particular, there is a one-one correspondence between $\mathfrak{M}_{A_e}\backslash\{A\}$ and $\mathfrak{M}_A$. Furthermore, there is a one-one correspondence between*

$$\mathrm{Epi}_C(A_e, Q)\backslash\{\eta_e: \ker(\eta_e) = A\}$$

and $\mathrm{Epi}_C(A, Q)$.

PROOF. Clearly A_e is a Q-bimodule. Let $I_e\in\mathfrak{I}_{A_e}$, and $I=I_e\cap A$. Since $I_e\not\subset A$, $I\neq A$ and I is a proper ideal of A. From Lemma 1.1 we know I_e is a Q-ideal. Let $q'\neq 0$ and $q'e+a'\in I_e$ (some such q' exists since $I_e\not\subset A$). We note $x(eq)=(xe)q=xq=(xq)e=x(qe)$, whence $x(eq-qe)=0$. Setting $x=e$, we see $eq=qe$. Thus $\sum_{i=1}^n q_{i1}q'q_{i2}e+\sum_{i=1}^n q_{i1}a'q_{i2}\in I_e$. Since Q is simple and $q'\neq 0$ we see $\{\sum_{i=1}^n q_{i1}q'q_{i2}\}=Q$, and hence for any q and some a, $qe+a\in I_e$. In particular, $-e_Qe+u=-e+u\in I_e$, for some $u\in A$. Direct calculation shows $ua-a$ and $au-a\in I$ for all $a\in A$. Hence I is a regular ideal of A. By Lemma 1.1, I is a Q-ideal. Conversely, suppose I is a regular ideal in A; let $u\in A$ be an identity modulo I. Set $I_e=\{x: x\in A_e;\ ux, xu\in I\}$. Since $u(e-u)=(e-u)u=u-u^2\in I$, whence $e-u\in I_e$, we have $I_e\not\subset A$. Moreover, $I_e\not\supset A_e$ since $e\in I_e$ implies $u\in I$, whence $a=(a-ua)+ua\in I$ for all $a\in A$, a contradiction. I_e is an ideal in A_e. In fact, if $x\in I_e$ and $y=qe+a\in A_e$, then $u(xy)=(ux)(qe+a)=(ux)q+(ux)a\in I$; and $(xy)u=x(qe+a)u=xqu+xau=[(xqu)-u(xqu)]+ux(qu)+ux(au)+[(xau)-u(xau)]\in I$, thus $xy\in I_e$. Similarly, $(yx)u$, $u(yx)\in I$, hence

$yx\in I_e$. Thus I_e is an ideal and by Lemma 1.1, I_e is a Q-ideal. Direct calculation shows $I_e\cap A=I$. In fact, if $a\in I_e\cap A$, then $ua\in I$, and hence $a=(a-ua)+ua\in I$. Conversely, if $a\in I$, then $ua=-(a-ua)+a\in I$, $au=-(a-au)+a\in I$, whence $a\in I_e\cap A$. It follows that the correspondence $\mathcal{I}_{A_e}\ni I_e\to I_e\cap A\in\mathcal{I}_A$ is epic. We show it is one-one. Suppose $I_e\cap A=J_e\cap A=I$ for I_e, $J_e\in\mathcal{I}_A$. As seen above, there exist u, $v\in A$ such that $-e+u\in I_e$, $-e+v\in J_e$. Then $-v+vu\in I$ and $-u+vu\in I$, whence $u-v\in I$. Suppose $qe+a\in I_e$. Then $uq+ua=u(qe+a)\in I$, $-a+ua=(-e+u)a\in I$, and consequently, also $uq+a=uq+ua-(-a+ua)\in I$. But then since $qe=eq$ and J_e, I are Q-ideals, $qe+a=(e-v)q+(u-v)q+uq+ua\in J_e$. Hence $I_e\subset J_e$. Similarly, we have $J_e\subset I_e$, and thus $I_e=J_e$. It follows that the correspondence $\mathcal{I}_{A_e}\ni I_e\to I_e\cap A=I\in\mathcal{I}_A$ is one-one.

Now, A is maximal in A_e and A_e/A is $\mathbf{C}$-isomorphic to Q. It follows easily from the above correspondence that there is a one-one correspondence between $\mathfrak{M}_{A_e}\setminus\{A\}$ and $\mathfrak{M}_A$. For if $M_e\in\mathfrak{M}_{A_e}\setminus\{A\}$, then $M_e\neq A$, and since A is maximal in A_e, $M_e\not\subset A$. Then $M_A=M_e\cap A$ is a maximal regular ideal in A. On the other hand, if $M_A\in\mathfrak{M}_A$ and $u\in A$ is such that $u/M_A=e_Q=$identity of $Q\cong A/M_A$; then $M_e\equiv\{x: x\in A_e,\ xu,\ ux\in M_A\}$ is a maximal ideal in A_e, $M_e\neq A$, and $M_e\cap A=M_A$.

We continue by letting $\eta\in\mathrm{Epi}_{\mathbf{C}}(A,Q)$, $\ker(\eta)=M_A$. Define $\eta_e\in\mathrm{Epi}_{\mathbf{C}}(A_e,Q)$ by the formula $\eta_e(qe+a)=\alpha_\eta(q)+\eta(a)$. Thus $\ker(\eta_e)$ is a maximal ideal M_e'. However, $M_e'\cap A=\{a: a\in A,\ \eta_e(a)=0\}=\ker(\eta)=M$. Thus $M_e'=M_e$ and A/M_e is $\mathbf{C}$-isomorphic to Q, whence A_e is Q-uniform.

If $\eta\in\mathrm{Epi}_{\mathbf{C}}(A, Q)$ let $\tau:\eta\to\eta_e\in\mathrm{Epi}_{\mathbf{C}}(A_e,Q)$ as defined above. Conversely, given $\eta_e\in\mathrm{Epi}_{\mathbf{C}}(A_e,Q)$ $(\ker(\eta_e)\neq A)$ let σ: $\eta_e\to\eta\in\mathrm{Epi}_{\mathbf{C}}(A,Q)$ where η is the restriction of η_e to A. We show that $\sigma\tau=$identity, $\tau\sigma=$identity. Indeed, $\tau\eta(qe+a)=\alpha_\eta(q)+\eta(a)$ and $(\sigma\tau\eta)$ $(a)=\eta(a)$. Furthermore, $\sigma\eta_e(a)=\eta(a)$, $(\tau\sigma\eta_e)(qe+a)=\alpha_\eta(q)+\eta(a)$. However, writing $u=e+m_e$, $m_e\in M_e$ as earlier, we find $qe=qu-qm_e$, $\alpha_{\eta_e}(q)=\eta_e(qe)=\eta_e(qu)=\eta(qu)=\alpha_\eta(q)$. Thus $(\tau\sigma\eta_e)(qe+a)=\alpha_{\eta_e}(q)+\eta(a)=\eta_e(qe+a)$.

In the course of the above we have established not only the one-one relationship between $\mathrm{Epi}_{\mathbf{C}}(A,Q)$ and $\mathrm{Epi}_{\mathbf{C}}(A_e,Q)\setminus\{\eta_e\colon \ker(\eta_e)=A\}$ but also the formula $\alpha_\eta=\alpha_{\eta_e}$ when $\tau(\eta)=\eta_e$.

In the following we topologize various sets as follows: $\mathrm{Epi}_{\mathbf{C}}(A,Q)$: the weak topology where a typical neighborhood is

$$N(\eta_0)=\{\eta: \|\eta(a_i)-\eta_0(a_i)\|_Q<\epsilon,\ a_i\in A,\ i=1,2,\cdots,n\}.$$

$\mathrm{Aut}_{\mathbf{C}}(Q)$: the weak topology where a typical neighborhood is

$$N(\alpha_0) = \{\alpha: \|\alpha(q_i) - \alpha_0(q_i)\|_Q < \epsilon,\ q_i \in Q,\ i = 1, 2, \cdots, n\}$$

$\mathfrak{M}_A$: the strongest topology such that the map

$$\rho: \mathrm{Epi}_C\,(A,Q) \ni \eta \to \ker(\eta) \in \mathfrak{M}_A$$

is continuous. Direct verification shows that the first two are Hausdorff topologies.

LEMMA 1.4. *Assume that for each $M_A \in \mathfrak{M}_A$ and u such that u/M_A $=$ identity of A/M_A the direct sum decomposition $A = Qu \oplus M$ obtains (cf. Lemma 1.2). Then there is a bijection θ of* $\mathrm{Epi}_C(A,Q)$ *onto* $\mathfrak{M}_A \times \mathrm{Aut}_C(Q)$. *Furthermore, if π is the projection of* $\mathfrak{M}_A \times \mathrm{Aut}_C(Q)$ *on* $\mathfrak{M}_A$ *then $\rho = \pi\theta$ is a continuous open map of* $\mathrm{Epi}_C(A,Q)$ *on* $\mathfrak{M}_A$.

PROOF. For $\eta \in \mathrm{Epi}_C(A,Q)$ let $\theta(\eta) = (\ker(\eta),\ \alpha_\eta) \equiv (M_A,\ \alpha_\eta)$. Here $\alpha_\eta(q) = \eta(qu)$ where $A = Qu \oplus M_A$. We show first that α_η is independent of the choice of u. Indeed if $u_1/M_A = u/M_A$, then $u_1 = u + m$, $m \in M_A$ and $\eta(qu_1) = \eta(qu) + \eta(qm)$. Since M_A is regular, it is a Q-ideal (Lemma 1.1) whence $\eta(qm) = 0$ and thus $\eta(qu) = \eta(qu_1)$.

The map θ is one-one since if $\theta(\eta_1) = \theta(\eta_2)$ then $\ker(\eta_1) = \ker(\eta_2) = M_A$ say. If $\eta_1 \neq \eta_2$ for some $qu+m$, $m \in M_A$, $\eta_1(qu+m) \neq \eta_2(qu+m)$ or $\alpha_{\eta_1}(q) \neq \alpha_{\eta_2}(q)$, contradicting $\theta(\eta_1) = \theta(\eta_2)$.

Let $\tilde{\theta}$: $\mathfrak{M}_A \times \mathrm{Aut}_C(Q) \ni (M_A,\ \alpha) \to \alpha\alpha_{\eta_0}^{-1}\eta_0 \in \mathrm{Epi}_C(A,Q)$ where η_0 is such that $\ker\,(\eta_0) = M_A$. We show $\tilde{\theta}$ is independent of the choice of η_0 and that $\theta\tilde{\theta} =$ identity (whence θ is surjective) and that $\tilde{\theta}\theta =$ identity (whence $\tilde{\theta}$ is surjective).

Indeed, if $\ker(\eta_1) = \ker(\eta_0) = M_A$ then for some $\bar{\alpha}$, $\eta_1 = \bar{\alpha}\eta_0$. Thus $\alpha_{\eta_1}^{-1}\eta_1 = \alpha_{\eta_1}^{-1}\bar{\alpha}\eta_0$. On the other hand, $\alpha_{\eta_1}(q) = \bar{\alpha}\eta_0(qu) = \bar{\alpha}\alpha_{\eta_0}(q)$, whence $\alpha_{\eta_1}^{-1} = \alpha_{\eta_0}^{-1}\bar{\alpha}^{-1}$ and so $\alpha_{\eta_1}^{-1}\eta_1 = \alpha_{\eta_0}^{-1}\bar{\alpha}^{-1}\bar{\alpha}\eta_0 = \alpha_{\eta_0}^{-1}\eta_0$.

Clearly $\ker(\alpha\alpha_{\eta_0}^{-1}\eta_0) = \ker(\eta_0)$ and $\alpha_{(\alpha\alpha_{\eta_0}^{-1}\eta_0)} = \alpha\alpha_{\eta_0}^{-1}\alpha_{\eta_0} = \alpha$. Thus $\theta\tilde{\theta} =$ identity. On the other hand $\tilde{\theta}(\ker(\eta),\ \alpha_\eta) = \alpha_\eta \cdot \alpha_\eta^{-1}\eta = \eta$. Thus $\tilde{\theta}\theta =$ identity.

By definition, ρ is continuous. We prove next that ρ is open. Thus let $V \subset \mathrm{Epi}_C(A,Q)$ be open. By virtue of the definitions of the topologies involved, to show $\rho(V)$ is open it suffices to show $\rho^{-1}(\rho(V))$ is open. However, $\eta' \in \rho^{-1}(\rho(V))$ iff $\eta' = \alpha\eta$ for some $\alpha \in \mathrm{Aut}_C(Q)$ and some $\eta \in V$. Thus $\rho^{-1}(\rho(V)) = \bigcup\{\alpha V: \alpha \in \mathrm{Aut}_C(Q)\}$. Hence we shall prove that αV is open and thereby establish that ρ is open.

We note that ρ_α: $\mathrm{Epi}_C(A,Q) \ni \eta \to \alpha\eta \in \mathrm{Epi}_C(A,Q)$ is one-one since if $\alpha\eta_1(a) = \alpha\eta_2(a)$ for all $a \in A$, then $\eta_1 = \eta_2$. Next ρ_α is continuous, since if

$$N(\rho_\alpha(\eta_0)) = \{\eta: \|\eta(a_i) - \rho_\alpha(\eta_0)(a_i)\|_Q < \epsilon,\ i = 1, 2, \cdots, n\}$$

let

$$N(\eta_0) = \{\eta : \|\eta(a_i) - \eta_0(a_i)\|_Q < \epsilon/\|\alpha\|,\ i = 1, 2, \cdots, n\}.$$

Then for $\eta \in N(\eta_0)$ we find $\rho_\alpha(\eta) \in N(\rho_\alpha(\eta_0))$. Hence ρ_α is continuous. Since $\rho_\alpha(\alpha^{-1}\eta) = \eta$ we see ρ_α is surjective and clearly $\rho_\alpha^{-1} = \rho_{\alpha^{-1}}$, whence ρ_α^{-1} is continuous. We conclude ρ_α is a self-homeomorphism (auteomorphism) of $\mathrm{Epi}_C(A,Q)$. Since $\rho_\alpha(V) = \alpha V$ we see αV is open and thus the map ρ is open.

LEMMA 1.5. *If A has an identity e_A the map $\tau : \eta \to \alpha_\eta$ is continuous in the topologies considered.*

PROOF. Let $\alpha_{\eta_0} = \tau(\eta_0)$ be given and let

$$N(\alpha_{\eta_0}) = \{\alpha : \|\alpha(q_i) - \alpha_{\eta_0}(q_i)\| < \epsilon,\ i = 1, 2, \cdots, n\}$$

be given. Let $U(\eta_0) = \{\eta : \|\eta(q_i e_A) - \eta_0(q_i e_A)\| < \epsilon,\ i = 1,\ 2, \cdots,\ n\}$. Then for $\eta \in U(\eta_0)$

$$\|\alpha_\eta(q_i) - \alpha_{\eta_0}(q_i)\| = \|\eta(q_i e_A) - \eta_0(q_i e_A)\| < \epsilon,$$

whence $\tau(U(\eta_0)) \subset N(\alpha_{\eta_0})$, and the continuity of τ is established.

COROLLARY. *If A has an identity e_A the map θ is continuous.*

PROOF. Indeed, $\theta(\eta) = (\rho(\eta),\ \tau(\eta))$ whence, since ρ and τ are continuous, θ is continuous.

Any general statement asserting the compactness of $\mathrm{Epi}_C(A,Q)$ is false. This follows from the fact, for some Q and A, e.g., $Q = \mathrm{End}_C(\mathbf{C}^n)$ $\equiv$ the set of linear endomorphisms of $\mathbf{C}^n$ and $A = C(X,\ Q)$ where X is compact Hausdorff, that the function $\hat{a}(\eta)$ may be unbounded, for some a, where $\hat{a}(\eta)$ is the function defined on $\mathrm{Epi}_C(A,Q)$ by the formula $\hat{a}(\eta) = \eta(a)$.

Indeed, $\mathrm{Aut}_C(\mathrm{End}_C(\mathbf{C}^n))$ is the set of inner automorphisms α_S given by

$$\alpha_S : q \to S^{-1}qS$$

and $S \in Gl(n,\ \mathbf{C})$. If $n = 2$ the following calculations show that one may choose S so that $\|\alpha_S\|$ (operator norm) is arbitrarily large (or small).

For $N > 0$, let numbers $u,\ v,\ w,\ x$ be chosen so that $uvwx \neq 0$ and let

$$q = \begin{pmatrix} u & v \\ w & x \end{pmatrix}.$$

Then if $\lambda\mu\neq 0$ and

$$S = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$$

we find

$$\alpha_S(q)\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} u \\ x \end{pmatrix} + \begin{pmatrix} v\mu\lambda^{-1} \\ w\mu^{-1}\lambda \end{pmatrix}.$$

Set $\lambda = 1$ and choose μ so that

$$\left\| \alpha_S(q)\begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\| > N\|q\| \left\| \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\|.$$

In this case $\|\alpha_S\| > N$. Similarly, if $q' = \alpha_S q$, then

$$\|\alpha_S^{-1} q'\|, \qquad \frac{\|\alpha_S^{-1} q'\|}{\|q'\|} = \frac{\|q\|}{\|q'\|} < \frac{\|q\|}{N\|q\|} = \frac{1}{N}.$$

Hence $\|\alpha_S^{-1}\| < 1/N$. Thus $0 = \inf\{\|\alpha_S\|\} < \sup\{\|\alpha_S\|\} = +\infty$. The set $\{q: \sup_{\alpha_S}\|\alpha_S q\| < +\infty\}$ must be of the first category and thus the set $\{q: \sup_{\alpha_S}\|\alpha_S q\| = +\infty\}$ is of the second category and nonempty. For a q such that $\sup_{\alpha_S}\|\alpha_S q\| = +\infty$, choose an $a_0 \in A$ and an η_0 such that $\eta_0(a_0) = q$. Then $\sup_\eta \|\hat{a}_0(\eta)\| \geqq \sup_{\alpha_S}\|\alpha_S q\| = +\infty$.

2. **Special cases.** If Q is a simple commutative Banach algebra with identity, then Q is isomorphic to $\mathbf{C}$. A parallel of this elementary fact is the

PROPOSITION 1. *If A is a Q-uniform semisimple Banach algebra and if A is not commutative then Q is not commutative.*

PROOF. Let $a_1, a_2 \in A$, $a_1a_2 - a_2a_1 \neq 0$. Then since A is semisimple, there is some regular maximal ideal M such that $a_1a_2 - a_2a_1 \notin M$, i.e.

$$a_1/M \cdot a_2/M - a_2/M \cdot a_1/M \neq 0.$$

If Q is commutative, the last relation cannot obtain.

COROLLARY. *If G is a locally compact group and if $L^1(G)$ is semi-simple (e.g., if G is abelian or compact) and Q-uniform, then G is abelian (whence $Q \cong \mathbf{C}$).*

PROOF. Define the map $h: L^1(G) \to \mathbf{C}$ by

$$h(f) = \int_G f dx.$$

Then $h \in \mathrm{Epi}_{\mathbf{C}}(L^1(G), \mathbf{C})$ and thus $L^1(G)/\ker(h) \cong \mathbf{C}$. Thus $Q \cong \mathbf{C}$, since we have assumed $L^1(G)$ is Q-uniform. Thus $L^1(G)$ must be commutative by Proposition 1 and therefore G is abelian.

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