ANNULUS CONJECTURE AND STABILITY OF HOMEOMORPHISMS IN INFINITE-DIMENSIONAL NORMED LINEAR SPACES

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Abstract. If $E$ is an arbitrary infinite-dimensional normed linear space, it is shown that if all homeomorphisms of $E$ onto itself are stable, then the annulus conjecture is true for $E$. As a result, this confirms that the annulus conjecture for Hilbert space is true. A partial converse is that for those spaces $E$ which have some hyperplane homeomorphic to $E$, if the annulus conjecture is true for $E$ and if all homeomorphisms of $E$ onto itself are isotopic to the identity, then all homeomorphisms of $E$ onto itself are stable.

Let $E$ denote a normed linear space with zero element $\theta$, and let $B_r(x)$ be the ball in $E$ of radius $r$ centered at $x$, $S_r(x) = \text{Bd } B_r(x)$, $B_r = B_r(\theta)$, and $S_r = S_r(\theta)$. $H(E)$ and $SH(E)$ will denote the homeomorphisms on $E$ and the stable homeomorphisms on $E$, respectively.

From [5] and [6] there is an inversion homeomorphism $i \in H(E)$ for $E$ infinite-dimensional such that $i(B_1) = E - \text{Int } B_1$, $i|S_1 = \text{identity}$, and $i^{-1} = i$. It can be seen in [7] and [8] that this homeomorphism is a useful tool in proving topological properties of infinite-dimensional normed linear spaces. In this paper $i$ will be used in establishing a type of engulfing theorem for infinite-dimensional $E$ which is somewhat analogous to Stallings' Engulfing Theorem in [9]. Brown and Gluck in [3] showed that $SH(E) = H^+(E)$ implies the annulus conjecture for finite-dimensional $E$, where $H^+(E)$ is the group of orientation preserving homeomorphisms of $E$. As an application of the engulfing theorem in this paper, it will be shown that $SH(E) = H(E)$ implies the annulus conjecture for infinite-dimensional $E$ (a slightly different technique than used in [3]). As a result, since Wong in [10] showed that $SH(l_2) = H(l_2)$, the annulus conjecture is true in Hilbert space and is consequently true in all infinite-dimensional separable Banach spaces, or more generally, in all infinite-dimensional separable Fréchet spaces (see [4] and [1]).

By a cell in $E$ is meant a subset $C$ of $E$ such that there exists a homeomorphism from the pair $(B_1, S_1)$ onto the pair $(C, \text{Bd } C)$. A
closed subset $K$ of $E - \text{Int } C$ is a **collar** of $C$ if there exists a homeomorphism $h$ from the pair $(B_2, B_1)$ onto the pair $(K \cup C, C)$ such that $h(S_2) = \text{Bd } (K \cup C)$. $C$ is **tame** if there is an $h \in H(E)$ such that $h(B_1) = C$. In [7] and [8] it is shown that $C$ is tame if and only if it has a collar. By a **sphere** in $E$ is meant a closed homeomorph of $S_1$ in $E$. A sphere is **tame** if it is the boundary of a tame cell. Sanderson showed in [8] that a sphere is tame if and only if it is bicollared. $A \subset E$ is an **annulus** if there exists a homeomorphism $h$ from $B_2 - \text{Int } B_1$ onto $A$ such that $\text{Bd } A = h(S_1 \cup S_2)$.

The annulus conjecture for $E$ is stated in [7] as follows.

$A_E$: If $C$ is a tame cell contained in $\text{Int } B_1$, then there exists a homeomorphism $h$ of $B_1$ onto itself such that $h(B_{1/2}) = C$ and $h|S_1 = \text{identity}$.

For infinite-dimensional $E$, it can be seen by using $i$ that $A_E$ is equivalent to the statement: the region between two disjoint tame spheres is an annulus.

$h \in H(E)$ is **stable** if it can be written as a finite composition of elements of $H(E)$ each of which is the identity on some open subset of $E$. The following is the stability conjecture for $E$.

$S_E$: $SH(E) = H(E)$ (for $E$ finite-dimensional $SH(E) = H^+(E)$).

**Lemma 1.** If $E$ is infinite-dimensional and $C$ is a collared cell with collar $K$, then there exists a collar $K' \subset K$ of $C$ and $g \in H(E)$ such that $g(C) = E - \text{Int } B_1$ and $g(C \cup K') = E - \text{Int } B_{1/2}$.

**Proof.** Let $h: (C \cup K, \text{Bd } (C \cup K)) \to (B_2, S_3)$ be a homeomorphism such that $h(C) = B_1$. Set $K' = h^{-1}(B_2 - \text{Int } B_1)$. By Lemma 6.1 of [7], there exists $f \in H(E)$ such that $f|K' \cup C = h|K' \cup C$. Let $k: B_2 - \text{Int } B_1 \to - \text{Int } B_{1/2}$ be a homeomorphism such that $k(S_2) = S_{1/2}$ and $k|S_1 = \text{identity}$. Define the homeomorphism $j': E - \text{Int } B_1 \to - \text{Int } B_{1/2}$ by letting $j'(B_2 - \text{Int } B_1) = k$ and, for $E - B_2$, letting $j'$ map $[x: \infty)$, $x \in B_2$, linearly onto $[k(x): \theta)$. Let $j: B_1 - \theta \to B_1$ be a homeomorphism such that $j|B_1 - \text{Int } B_{1/2} = \text{identity}$. Define $g' \in H(E)$ by $g'(x) = i(x)$ if $x \in B_1$, and $g'(x) = jf'(x)$ if $x \in E - B_1$. Finally define $g \in H(E)$ by $g = g'f$.

**Theorem 1 (Engulfing Theorem).** Let $E$ be infinite-dimensional, let $A \subset E$, let $C$ be a collared cell in $E$ with collar $K$ such that $(C \cup K) \cap A = \emptyset$, and let $U \subset E$ be open such that $[E - (C \cup K)] \cap U \neq \emptyset$. Then there exists $h \in H(E)$ such that $A \subset h(U)$ and $h|C = \text{identity}$.

**Proof.** By Lemma 1, let $K' \subset K$ be a collar of $C$ and $g \in H(E)$ be such that $g(C) = E - \text{Int } B_1$ and $g(C \cup K') = E - \text{Int } B_{1/2}$. Choose $B_\gamma(x) \subset g(U) \cap \text{Int } B_{1/2}$. Let $f \in H(E)$ be such that $B_{1/2} = f[B_\gamma(x)]$ and
Using a proof very similar to the proof of the above theorem, the following result concerning the fattening of a collar can be established.

**Theorem 2.** Let \( E \) be infinite-dimensional, let \( A \subseteq E \), and let \( C \) be a collared cell in \( E \) with collar \( K \) such that \( C \cup \operatorname{Cl} A \neq E \). Then there exists \( h \in H(E) \) such that \( A \subseteq h(C \cup K) \) and \( h|_C = \text{identity} \).

**Proof.** By Lemma 1, let \( K' \subseteq K \) be a collar of \( C \) and \( g \in H(E) \) be such that \( g(C) = E - \operatorname{Int} B_1 \) and \( g(C \cup K') = E - \operatorname{Int} B_{1/2} \). Choose \( B_\gamma(x) \subseteq \operatorname{Int} B_1 - g(A) \). Let \( f \in H(E) \) be such that \( B_{1/2} = f[B_\gamma(x)] \) and \( f|_{E - B_1} = \text{identity} \). Define \( h \in H(E) \) by \( h = g^{-1}f|_{E - B_1} \).

**Lemma 2.** Let \( A \) be a nondense subset of \( E \), and let \( f \in H(E) \) and \( U \subseteq A \) be open such that \( f|_U = \text{identity} \). Then there exist \( f_1, f_2, f_3 \in H(E) \) and \( U_1, U_2, U_3 \) open subsets of \( E \) such that \( U_1 \subseteq E - A, U_2 \subseteq E - f_1(A), U_3 \subseteq E - f_2f_1(A) \), \( f = f_3f_2f_1 \), and \( f_i|_{U_i} = \text{identity} \) for \( i = 1, 2, 3 \).

**Proof.** Let \( B_\gamma(x) \subseteq U \) and \( B_\delta(y) \subseteq E - A \). Let \( N > M > 0 \) be such that \( \operatorname{Int} B_\gamma(y) \cap \operatorname{Int} B_M(x) \neq \emptyset \) and \( \operatorname{Int} B_\delta(y) \cap [E - B_N(x)] = \emptyset \). Define \( f_1 \in H(E) \) such that \( f_1[B_M(x)] = B_\gamma(x) \) and \( f_1[E - B_N(x)] = \text{identity} \). Define \( f_2, f_3 \in H(E) \) by \( f_2 = f \) and \( f_3(x) = f_2f_1^{-1}f^{-1}(x) \) if \( x \in [B_N(x)] \), and \( f_3(x) = x \) if \( x \in E - f[B_N(x)] \). Choose \( U_1 \) open in \( B_\gamma(y) - B_N(x) \), \( U_2 \) open in \( B_\gamma(x) \cap f_1[B_\delta(y)] \), and \( U_3 \) open in \( f[B_\delta(y)] - B_N(x) \).

**Lemma 3.** Let \( f \in H(E) \), and \( U \subseteq A \) be open such that \( f|_U = \text{identity} \). Let \( k \) be such that \( 1 \leq k \leq m \). If \( V_k \subseteq [E - g_{k-1} \cdots g_1g_0(B_1)] \neq \emptyset \) (\( g_0 = \text{identity} \)), then choose \( U_{k1} \) open in \( V_k \cap [E - g_{k-1} \cdots g_1g_0(B_1)] \) and let \( f_{k1} = g_k \). If \( V_k \subseteq [E - g_{k-1} \cdots g_1g_0(B_1)] = \emptyset \), then by Lemma 2, there exist \( f_{k1}, f_{k2}, f_{k3} \in H(E) \), open \( U_{k1} \subseteq E - g_{k-1} \cdots g_1g_0(B_1) \), open \( U_{k2} \subseteq E - f_{k1}g_{k-1} \cdots g_1g_0(B_1) \), and open \( U_{k3} \subseteq E - f_{k2}f_{k1}g_{k-1} \cdots g_1g_0(B_1) \) such that \( g_k = f_{k3}f_{k2}f_{k1} \) and \( f_{ki}|_{U_{ki}} = \text{identity} \), \( i = 1, 2, 3 \). The \( f_{ij} \) and \( U_{ij} \) can be relabeled to give the desired \( f \) and \( U_i, i = 1, \cdots, n \).

**Addendum to Lemma 3.** In Lemma 3 the \( f_i \) can further be chosen so that \( f_i \cdots f_1(B_1) \subseteq f_{i+1} \cdots f_1(B_1) \) for \( i = 1, \cdots, n-1 \).

**Proof.** Let \( U'_n = U_n \) and \( f'_n = f_n \). For \( k \) such that \( 1 \leq k \leq n-1 \), define \( U'_{n-k}, f'_{n-k}, U''_{n-k+1} \), and \( f''_{n-k+1} \) by an inductive step as follows.

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Choose \( B_\gamma(x) \subset f_i^{-1} \cdots f_{n-k}^{-1} f_{n-k+1}^{-1} f_1(B_1) \), \( B_\delta(y) \subset f_i^{-1} \cdots f_{n-k}^{-1} (U_{n-k+1}) \) such that \( B_\gamma(x) \cap (B_\delta(y) \cup B_\varepsilon(z)) = \emptyset \). Let \( g \in H(E) \) be such that \( g(B_1) = B_\gamma(x) \) and \( g| (B_\delta(y) \cup B_\varepsilon(z)) = \text{identity} \). Define \( U_{n-k} = f_{n-k} \cdots f_1(\text{Int } B_\delta(y)) \), \( f_{n-k} = f_{n-k} \cdots f_1 g f_{n-k} \cdots f_1(\text{Int } B_\varepsilon(z)) \), and \( f'_{n-k+1} = f'_{n-k+1} f_{n-k} f'_{n-k}^{-1} \). Finally let \( U_i'' = U_i' \) and \( f_i'' = f_i' \). The \( f_i' \) and \( U_i'' \) are then the desired homeomorphisms and open sets.

**Theorem 3.** If \( E \) is infinite-dimensional and if \( f \in SH(E) \) such that \( f(B_1) \) is bounded, then there exists \( N > 0 \) and \( g \in H(E) \) such that \( g|B_1 = f|B_1 \) and \( g|(E-B_N) = \text{identity} \).

**Proof.** By Lemma 3 and its addendum, there exists, for \( i = 1, \ldots, n, f_i \in H(E) \) and open \( U_i \subset E - f_{i-1} \cdots f_1 f_0(B_1) \) (\( f_0 = \text{identity} \)) such that \( f_i|U_i = \text{identity} \), \( f = f_n \cdots f_1 \), and for \( i = 1, \ldots, n - 1 \), \( f_i' \cdots f_0(B_1) \subset f_{i+1} \cdots f_1 f_0(B_1) \). Without loss of generality it can be assumed that each \( U_i \) is bounded. Choose \( N > 0 \) such that \( B_1 \cup f(B_1) \cup [U_{n-1} U_i] \subset B_N \). Choose \( B_\gamma(x) \subset f_1(B_1) \) and \( B_\delta(y) \subset U_1 \). Let \( g' \in H(E) \) be such that \( g'[B_\gamma(x)] = B_1 \) and \( g'|[B_\delta(y) \cup (E-B_N)] = \text{identity} \). Redefine \( f_1 \) to be \( f_1 g' \). For each \( i = 1, \ldots, n \), by the Engulfing Theorem there exists \( g_i \in H(E) \) such that \( E-B_N \subset g_i(U_i) \) and \( g_i|f_i' \cdots f_i[B_\gamma(x)] = \text{identity} \). Define \( f_i'' = g_i f_i g_i^{-1} \). Then define the desired \( g \) by \( g = f_n'' \cdots f_1'' (g')^{-1} \).

**Theorem 4.** \( S_E \) implies \( A_E \).

**Proof (For \( E \) infinite-dimensional).** Let \( C \) be a tame cell contained in \( \text{Int } B_1 \). By the Half-open Annulus Theorem [7], there exists a homeomorphism \( f \) of \( \text{Int } B_1 \) onto itself such that \( f(B_{1/2}) = C \). By a slight modification of Lemma 3.1 in [7], there exists a cell \( C' \subset B_1 \) such that \( C \cup B_{1/2} \subset C' \) and \( B_1 - \text{Int } C' \) is a collar of \( C' \). Define \( g : \text{Int } B_1 \rightarrow E \) radially in the following manner. Set \( S' = \text{Bd } C' \). For \( x \in E \), because of the construction of \( C' \) in Lemma 3.1 in [7], \( (\text{Ray } [\theta : x]) \cap S' \) is a single point—call it \( x_1 \). Also \( (\text{Ray } [\theta : x]) \cap S_1 = x_2 \). Let \( g \) map \([x_1 : x_2] \) linearly onto \([x_1 : \infty]\) and let \( g|C' = \text{identity} \). Set \( f = g f g^{-1} \). Then since \( S_E \) is true, \( f \in SH(E) \). Hence by Theorem 3 there exist \( N > 0 \) and \( g \in H(E) \) such that \( g|B_1 = f|B_1 \) and \( g|(E-B_N) = \text{identity} \). Let \( h : \text{Int } B_1 \rightarrow \text{Int } B_1 \) be defined by \( h = g^{-1} g \). Finally \( h \) can be extended to the identity on \( S_1 \).

**Corollary 1.** \( A_E \) is true in all separable infinite-dimensional Fréchet spaces.

The annulus conjecture for the Hilbert cube has been observed to be true by Anderson in Corollary 10.6 of [2]. His proof of this also
follows from the fact that homeomorphisms of the Hilbert cube onto itself are stable, which he shows in that paper.

Let $HI(E)$ denote the homeomorphisms on $E$ which are isotopic to the identity. $I_E$ will be the conjecture that $HI(E) = H(E)$ (use orientation preserving homeomorphisms for $E$ finite-dimensional).

The following theorem is well known and can be found for example proved for the special spaces of $E^n$, $s$, and $I^n$ in [3], [10], and [1], respectively.

**Theorem 5.** $S_E$ implies $I_E$.

**Proof.** Let $h \in SH(E)$.

**Case 1.** Assume that $h$ is fixed on $B_1$. Define

$$H_t(x) = \frac{(1 + t)}{(1 - t)}h[(1 - t)/(1 + t)x],$$

for $0 \leq t < 1$, and $H_t(x) = x$, for $t = 1$. Let $x \in E$ and $\gamma > 0$, and choose $\delta = \max\left[\frac{||x|| + \gamma - 1}{||x|| + \gamma + 1}, 0\right]$. Then for any $y \in B_\gamma(x)$ and $t \in (0, 1)$, $H_t(y) = y$. Thus $H_t$ is an isotopy from $h$ to the identity.

**Case 2.** $h = h_n \cdots h_1$, where each $h_i$ is fixed on some $B_{\gamma_i}(x_i)$. Then as in the above case define isotopies $H^i_t$ from $h_i$ to the identity. Finally define the isotopy $H = H_n \cdots H_1$ from $h$ to the identity.

**Lemma 4.** Let $A_E$ hold, let $C$ be a tame cell, and let $B_\gamma(x) \subset \text{Int } C$. Then there exists $g \in H(E)$ such that $g(C) = B_{\gamma+1}(x)$ and $g|B_\gamma(x) = \text{identity}$.

**Proof.** Let $h \in H(E)$ be such that $h(B_1) = C$. By $A_E$, let $f \in H(E)$ be such that $f(B_{1/2}) = h^{-1}(B_\gamma(x))$ and $f|(E - B_1) = \text{identity}$. Define $j \in H(E)$ so that $j(B_{1/2}) = B_\gamma(x)$ and $j(B_1) = B_{\gamma+1}(x)$. Let $p^*_{\gamma}$ be the projection of $E - \theta$ onto $S_\gamma(x)$ along the rays emanating from $x$. Define $k \in H(E)$ so that $k|B_\gamma(x) = hj^{-1}|B_\gamma(x)$, and for $y \in S_\gamma$, $r > \gamma$, $k(y) = p^*_{\gamma} k p^*_{\gamma}(y)$. Then set $g = kjf^{-1}h^{-1}$.

**Theorem 6.** $A_E$ and $I_E$ together imply $S_E$ for all normed linear spaces $E$ which contain a hyperplane homeomorphic to $E$.

**Proof.** Let $h \in H(E)$, and let $B_\gamma(x) \subset h(\text{Int } B_1)$. Then by Lemma 4, there exists $g \in H(E)$ such that $gh(B_1) = B_{\gamma+1}(x)$ and $g|B_\gamma(x) = \text{identity}$. Let $k \in H(E)$ be such that $k(B_{\gamma+1}(x)) = B_1$ and $k|(E - B_N) = \text{identity}$ for some $N$. In [6] it is seen that all hyperplanes of $E$ are homeomorphic to $S_1$ in $E$. Then because $I_E$ holds, there exists an isotopy $F_t: S_1 \rightarrow S_1$, $0 \leq t \leq 1$, such that $F_0 = kgh|S_1$ and $F_1 = \text{identity}$. Let $p_\gamma$, $r > 0$, be the radial projection of $E - \theta$ onto $S_\gamma$. Then define $f \in H(E)$ as follows. For $x \in S_\gamma$, $1 \leq r \leq 2$, let $f(x) = p_\gamma F_{r-1}p_1(x), f|B_1$
\[ = kgh | B_1, \text{ and } f | (E - B_2) = \text{identity}. \] Then \( h = g^{-1}k^{-1}f(f^{-1}kgh) \), which is stable.

**Corollary 2.** \( A_E \) and \( I_E \) together are equivalent to \( S_E \) for all normed linear spaces \( E \) which contain a hyperplane homeomorphic to \( E \).

Similar arguments give the finite-dimensional result found in [3].

**Theorem 7.** \( A_n \) and \( I_{n-1} \) together are equivalent to \( S_n \).

**References**


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