A RESULT OF BASS ON CYCLOTOMIC EXTENSION FIELDS

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In [1] Bass stated the result given below as Proposition 1 and derived some consequences. His proof of the proposition itself, however, contains a gap; Lemmas 2 and 3 are false as stated. The purpose of this note is to fill the gap by proving the slightly stronger Proposition 2.

We retain the notation of [1]. In particular $k_m = k(\zeta_m)$ where $\zeta_m = e^{2\pi i/m}$. The letters $m, n, a, b, c, d, r, s, t, u, v$ will denote nonnegative integers, $p$ is a prime integer, and $K = k(i)$.

**Proposition 1.** Given $k$ and $n$, there is an $m$ such that $k_m^* \cap k^* \subset k^*$.

**Proposition 2.** Given $k$ there is an $m$ such that for all $n$, $k_m^* \cap k^* \subset k^*$.

**Lemma 1.** Suppose $\zeta \in k$ if $p = 2$. Then if $r = p^a$, $k_r^* \cap k^* \subset k^*$. (For proof see p. 39 of [2].)

**Lemma 2.** Given $p$ and $k$ with $\zeta \in k$ if $p = 2$, suppose $r = p^a$ and $v$ are such that $t^{pr} \in k_v$. Then for all $t = p^c$, $k_r^* \cap k^* \subset k^*$.

**Proof.** If $c = 0$ the result is trivial; assume $c > 0$.

*Case 1.* $\zeta_p \in k$ or $\zeta_p \notin k_v$.

For any $u = p^d$, $d > 0$, any $r$th power, $z \in k^*$ of an element in $k_v^*$ is a $p$th power of an element in $k_v^*$. If not, $X^{ru} - z$ would be irreducible over $k$ [3, p. 221], hence all its roots would lie in $k_v$, which is normal over $k$, hence $\zeta_u \notin k_v$, contrary to supposition.

Therefore, if $x = y^r, x \in k$, $y \in k_v^*$, then $x = w^p, w \in k_v^*$, and $w^{-1} y^{r/p}$ is a $p$th root of 1 in $k_v$, hence in $k$, and $y^{r/p} \in k$. Repeating the argument if necessary we conclude, $y^r \in k_v^*, x = y^r \in k_v^*$.  

*Case 2.* $\zeta_p \notin k, \zeta_p \in k_v$.

If $x \in k$ is an $r$th power of something in $k_v$, then by Case 1, $x$ is a $t$th power of something in $k_v \subset k_v$. Taking norms from $k_v$ to $k$ and noting that $[k_p : k]$ is prime to $t$ gives the result.

**Lemma 3.** Let $s = 2^b$ be such that $\zeta_{2^b} \notin K$. Then for any $t = 2^c$, $K^* \cap k^* \subset k^*$.

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Proof (Following [2]). Let \( x = y^{st}, x \in k^*, y \in K^* \). If \( y \in k^* \) there is nothing to prove, so assume \( u = 2^d \) such that \( y^u \in k^*, y^{2u} \in k^* \). Then \( y^u = iz, z \in k^* \) and if \( \sigma \) denotes conjugation over \( k \), \( (y^{-1}y^\sigma)^u = -1 \). Hence \( u < s, y^u \in k^* \), \( x = y^{st} \in k^{st} \).

We are now ready to prove Proposition 2. For all ramified odd \( p \) let \( a_p \) denote one plus the exponent of \( p \) in the ramification degree, from \( Q \) to \( k \), of some prime dividing \( p \); for unramified odd \( p \) let \( a_p = 0 \), and let \( a_2 \) be one plus the exponent of 2 in the ramification degree, from \( Q \) to \( K \), of some prime dividing 2. Let \( r_p = p^{-a_p} \). Then for all \( p \) and \( v \) prime to \( p \), \( \zeta_{pr_p} \in K_v \), in fact \( \zeta_{s_{pr_p}} \in K_v \). Let \( s_p = r_p \) for \( p \) odd and \( s_2 = r_2^2 \), and let \( m = \prod s_p \). Then for any \( n = \Pi t_p, t_p = p^{-a_p} \), letting \( u_p = s_p t_p \),

\[
\begin{align*}
k^{*n} \cap k^* & = \left( \bigcap_p k^{*n/p} \right) \cap k^* \\
& \subseteq \left( \bigcap_{p \neq 2} k^{*n/p}_m \cap k^{*n/u_p} \right) \cap (K^{*u_2}_m \cap K_{mn/u_2}) \cap k^* \\
& \subseteq \left( \bigcap_{p \neq 2} k^{*n/p}_m \cap (K^{*u_2}_m) \cap k^* \right) \quad \text{(by Lemma 1)} \\
& \subseteq \left( \bigcap_{p \neq 2} k^{*n/p}_m \cap (K^{*u_2}_m) \cap k^* \right) \quad \text{(by Lemma 2)} \\
& \subseteq \bigcap_{p} (k^{*n/p}_m) = k^{*n} \quad \text{(by Lemma 3)}.
\end{align*}
\]

Proposition 3. If \( E = 2D_{k/Q} \), then \( \bigcap k^{*E_{s_{pr_p}}^r} = \{1\} \).

Proof. \( k \) contains no nontrivial roots of unity of order prime to \( E \). Hence if \( x \in k^*, x \neq 1, x \in k^{*s} \) for some \( s = E^b \). The only odd primes in the \( m \) of Proposition 2 are ramified ones, hence \( m \mid t \) for some \( t = E^c \). Then \( x \in k^{*t} \cap k^* \subset k^{*m/l} \subset k^{*n} \).

References


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