A RESULT OF BASS ON CYCLOTOMIC
EXTENSION FIELDS

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In [1] Bass stated the result given below as Proposition 1 and
derived some consequences. His proof of the proposition itself, how-
ever, contains a gap; Lemmas 2 and 3 are false as stated. The purpose
of this note is to fill the gap by proving the slightly stronger Proposition
2.

We retain the notation of [1]. In particular \( k_m = k(\zeta_m) \) where
\( \zeta_m = e^{2\pi i/m} \). The letters \( m, n, a, b, c, d, r, s, t, u, v \) will denote nonnega-
tive integers, \( p \) is a prime integer, and \( K = k(i) \).

**Proposition 1.** Given \( k \) and \( n \), there is an \( m \) such that \( k_m \cap k \supseteq k^n \).

**Proposition 2.** Given \( k \) there is an \( m \) such that for all \( n \), \( k_m \cap k \supseteq k^n \).

**Lemma 1.** Suppose \( \zeta_p \in k \) if \( p = 2 \). Then if \( r = p^a \), \( k_r \cap k \supseteq k^r \). (For
proof see p. 39 of [2].)

**Lemma 2.** Given \( p \) and \( k \) with \( \zeta_p \in k \) if \( p = 2 \), suppose \( r = p^a \) and \( v \) are
such that \( t^v \in k \). Then for all \( t = p^c \), \( k_v \cap k \supseteq k^v \).

**Proof.** If \( c = 0 \) the result is trivial; assume \( c > 0 \).

**Case 1.** \( \zeta_p \in k \) or \( \zeta_p \in k_v \).

For any \( u = p^d, d \geq 0 \), any \( r \)th power, \( z \in k \) of an element in \( k_r \)
is a \( p \)th power of an element in \( k_r \). If not, \( X^r - z \) would be irreducible
over \( k \) [3, p. 221], hence all its roots would lie in \( k_v \), which is normal
over \( k \), hence \( \zeta_r \in k_v \), contrary to supposition.

Therefore, if \( x = y^t, x \in k, y \in k_r \), then \( x = w^p, w \in k \), and \( w^{-1}y^t \) is
a \( p \)th root of 1 in \( k_v \), hence in \( k \), and \( y^t \in k \). Repeating the argument
if necessary we conclude, \( y^t \in k \), \( x = y^t \in k \).

**Case 2.** \( \zeta_p \in k \), \( \zeta_p \in k_v \).

If \( x \in k \) is an \( r \)th power of something in \( k_r \) then by Case 1, \( x \) is a
\( t \)th power of something in \( k_r \). Taking norms from \( k_p \) to \( k \)
and noting that \( [k_p : k] \) is prime to \( t \) gives the result.

**Lemma 3.** Let \( s = 2^b \) be such that \( \zeta_{2^b} \in K \). Then for any \( t = 2^c \),
\( K^{2^b} \cap k \supseteq k^t \).

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Proof (Following [2]). Let \( x = y^u, x \in \mathbb{k}^*, y \in \mathbb{K}^* \). If \( y \in \mathbb{k}^* \) there is nothing to prove, so assume \( u = 2^d \) such that \( y^u \in \mathbb{k}^*, y^{2u} \in \mathbb{k}^* \). Then \( y^u = iz, z \in \mathbb{k}^* \) and if \( \sigma \) denotes conjugation over \( k, (y^{-1}y^\sigma)^u = -1 \). Hence \( u < s, y^u \in \mathbb{k}^*, x = y^u \in \mathbb{k}^{*t} \).

We are now ready to prove Proposition 2. For all ramified odd \( p \) let \( a_p \) denote one plus the exponent of \( p \) in the ramification degree, from \( \mathbb{Q} \) to \( \mathbb{k} \), of some prime dividing \( p \); for unramified odd \( p \) let \( a_p = 0 \), and let \( a_2 \) be one plus the exponent of 2 in the ramification degree, from \( \mathbb{Q} \) to \( \mathbb{K} \), of some prime dividing 2. Let \( r_p = p^{a_p} \). Then for all \( p \) and \( v \) prime to \( p \), \( \zeta_{p^v} \in \mathbb{k}^v \), in fact \( \zeta_{p^v} \in \mathbb{K}^v \). Let \( s_p = r_p \) for \( p \) odd and \( s_2 = r_2^2 \), and let \( m = \prod s_p \). Then for any \( n = \prod t_p, t_p = p^{c_p} \), letting \( u_p = s_p t_p \),

\[
\mathbb{k}^{*mn} \cap \mathbb{k}^* = \left( \bigcap_p \mathbb{k}^{*u_p} \right) \cap \mathbb{k}^*
\subset \left( \bigcap_{p 
eq 2} \mathbb{k}^{*u_p} \right) \cap \left( \mathbb{k}^{*u_2} \cap \mathbb{k}^{*u_2/u_2} \right) \cap \mathbb{k}^* \tag{by Lemma 1}
\subset \left( \bigcap_{p 
eq 2} \mathbb{k}^{*u_p} \right) \cap \left( \mathbb{k}^{*u_2} \cap \mathbb{k}^{*u_2/u_2} \right) \cap \mathbb{k}^* \tag{by Lemma 2}
\subset \bigcap_{p \neq 2} \mathbb{k}^{*u_p} \cap \mathbb{k}^{*u_p} \cap \mathbb{k}^* \tag{by Lemma 3}.
\]

Proposition 3. If \( E = 2D_{\mathbb{k}/\mathbb{Q}} \), then \( \bigcap \mathbb{k}^{*E} = \{1\} \).

Proof. \( k \) contains no nontrivial roots of unity of order prime to \( E \). Hence if \( x \in \mathbb{k}^*, x \neq 1, x \in \mathbb{k}^{*s} \) for some \( s = E^b \). The only odd primes in the \( m \) of Proposition 2 are ramified ones, hence \( m \mid t \) for some \( t = E^c \). Then \( x \in \mathbb{k}^{*t} \cap \mathbb{k}^* \subset \mathbb{k}^{*t/m} \subset \mathbb{k}^{*s} \).

References