A NOTE ON MOUFANG VEBLEN-WEDDERBURN SYSTEMS

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Abstract. The purpose of this note is to show that a Veblen-Wedderburn system with multiplicative Moufang identity is a near field if its dimension $d$ over its kern does not exceed 7.

1. Introduction. In a recent paper [2] Kallaher investigates (left) Veblen-Wedderburn systems (and the corresponding projective planes) in which the Moufang identity

$$ (x \cdot y) \cdot (z \cdot x) = (x \cdot (y \cdot z)) \cdot x $$

holds. Such a system is called a Moufang (left) Veblen-Wedderburn system (MVW system). Fields, near fields and Cayley-Dickson algebras are examples of MVW systems. A proper MVW system is one in which the other distributive law does not hold. The only proper MVW systems known are the near fields. Kallaher [2] obtains two sets of conditions under which an MVW system is a near field. Recently the author has been able to show [3] that there are no proper finite MVW systems other than the near fields. The object of this note is to extend this result to the infinite systems of dimension $d \leq 7$ over their kerns.

2. During the course of the proof of Theorem 1 the following results are needed. Proofs of these results may be found in the references indicated.

Result 1. Every two elements of a Moufang loop generates a subgroup (di-associativity) (Bruck [1]).

Result 2. Let $F(\cdot, \cdot)$ be an MVW system with kern $K$. If $A(\cdot)$ is a maximal associative subloop contained in the loop $F'(\cdot)$, then $B(\cdot, \cdot)$ is a maximal near field contained in $F(+, \cdot)$ where $B = A \cup \{0\}$ and $F'$ consists of all nonzero elements from $F$. Further $B$ contains $K$ [3, Lemmas 3.1 and 3.2].

Theorem 1. Let $F(\cdot, \cdot)$ be a left MVW system of dimension $d$ over its kern $K$ (as a right vector space). If $d \leq 7$, then $F(+, \cdot)$ is a near field.

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Proof. In the course of this proof we use freely the right inverse property, left distributive law and some properties of the kern \( K \). Also we write \( ab \) in place of \( a \cdot b \). If \( d = 1 \), then \( F = K \) and the theorem is obvious. Suppose \( 1 < d \leq 7 \). Let \( x \) be an element from \( F \) which does not belong to \( K \) and \( G = \langle x \rangle \) be the subloop of \( F'(\cdot) \) generated by \( x \). \( G \) is obviously an associative subloop of \( F'(\cdot) \) and consequently there exists a maximal associative subloop \( A \) in \( F'(\cdot) \) which contains \( G \). From Result 2 it follows that \( B(\cdot) \) is a maximal near field where \( B = A \cup \{0\} \). If \( B = F \), the theorem is proved. Suppose \( B < F \). Then there is an element \( y \) in \( F \) such that \( y \in B \). Let \( H = \langle x, y \rangle \), the subloop of \( F'(\cdot) \) generated by \( x \) and \( y \) (\( H \) exists since \( F'(\cdot) \) is di-associative). Let \( M \) be a maximal associative subloop of \( F'(\cdot) \) containing \( H \). Using Result 2 again we obtain that \( N(\cdot) \) is a near field where \( N = M \cup \{0\} \). We claim that \( N = F \) and consequently \( F(\cdot) \) is a near field. Suppose \( N < F \). Then there is an element \( z \) in \( F \) which does not belong to \( N \). We now show that the existence of \( z \) leads to the conclusion that the set \( T = \{1, x, y, xy, z, zx, zy, z(yx)\} \) is independent over \( K \) implying a contradiction that \( F(\cdot) \) is of dimension \( d \geq 8 \) over its kern. Firstly we show that the set \( \{1, x, y, xy\} \) is independent over \( K \). Suppose there are elements \( a, b, c \) and \( k \) in \( K \) such that \( a + xb + yc + yxk = 0 \). Then it follows that \( y(c + xk) = -(a + xb) \). Suppose \( c + xk \neq 0 \). Then \( y = -(a + xb)(c + xk)^{-1} \in B \) a contradiction to the choice of \( y \). Thus \( a + xb = 0 \) and consequently \( c + xk = 0 \) which imply that \( a = b = c = d = 0 \). Hence the set \( \{1, x, y, xy\} \) is independent over \( K \).

Suppose there are elements \( a, b, c, k, e, f, g \) and \( h \) in \( K \) such that \( a + xb + yc + yxk + ze + xzf + zyg + z(yx)h = 0 \). This relation may be rewritten as \( 2X = Y \) where \( X = (e + xf + yg + yxh) \) and \( Y = -(a + xb + yc + yxk) \). Suppose \( X \neq 0 \). Then \( z = YX^{-1} \in N \) a contradiction to the choice of \( z \). Hence \( X = 0 \) and consequently \( Y = 0 \). Since the set \( \{1, x, y, xy\} \) is independent over \( K \), we obtain that \( a = b = c = k = e = f = g = h = 0 \). Thus \( T \) is an independent set over \( K \). This completes the proof of the theorem.

The question of existence of infinite proper MVW systems of dimension \( d \) over their kerns for \( d \geq 8 \) still remains unresolved.

References


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