COMMUTATIVITY AND COMMON FIXED POINTS IN RECURSION THEORY

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Let N be the set of nonnegative integers and, for $e \in N$, let ϕ_e be the partial recursive function of one argument having index e. In 1938 [1, The Recursion Theorem] Kleene showed that if f is any recursive function then, for some number c, $\phi_c \simeq \phi_{f(c)}$. It follows that if W_e (the recursively enumerable (r.e.) set with index e) is defined as the domain of ϕ_e, then $W_c = W_{f(c)}$. Call a number-theoretic function h *well-defined on the r.e. sets* if, for all $m, n \in N$, $W_m = W_n \rightarrow W_{h(m)} = W_{h(n)}$. In this paper we show that if f, g are recursive functions which are well-defined on the r.e. sets and which commute as maps of the r.e. sets (i.e., for all $n \in N$, $W_{f(g(n))} = W_{g(f(n))}$), then they have a common fixed point (i.e., for some $e \in N$, $W_e = W_{f(e)} = W_{g(e)}$). We also give an example which shows that the assumption of well-definedness cannot be eliminated.

First we prove a lemma related to the Myhill-Shepherdson Theorem [2, p. 359, Theorem XXIX (6)]. From now on, whenever f is well defined on the r.e. sets and W is an r.e. set, we shall write $f(W)$ for $W_{f(e)}$ where e is any number such that $W = W_e$.

LEMMA. *If f is a partial recursive function well-defined on the r.e. sets and W is an r.e. set, then*

$$f(W) = \bigcup \{ f(F) \mid F \subseteq W \;\&\; F \text{ is finite} \}.$$

PROOF. Our proof consists of two applications of Kleene's Recursion Theorem.

$f(W) \subseteq \bigcup \{ f(F) \mid F \subseteq W \}$: Suppose $n \in f(W)$. Let h be a recursive function such that, for all x,

$$\begin{aligned} W_{h(x)} &= W \quad \text{if } n \notin f(W_x), \\ &= \text{some finite subset } F \text{ of } W \quad \text{if } n \in f(W_x), \end{aligned}$$

and choose c satisfying $W_c = W_{h(c)}$. Clearly, $W_c \subseteq W$. Suppose $n \notin f(W_c)$. Then $W_c = W$, a contradiction, since $n \in f(W)$. So $n \in f(W_c)$. But then W_c is finite, so $n \in \bigcup \{ f(F) \mid F \subseteq W \}$.

$\bigcup \{ f(F) \mid F \subseteq W \} \subseteq f(W)$: Suppose $F \subseteq W$ and $n \in f(F)$. Let h be a recursive function such that, for all x,

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$$W_{h(x)} = F \quad \text{if } n \notin f(W_x),$$
$$\quad\; = W \quad \text{if } n \in f(W_x),$$

and choose c satisfying $W_c = W_{h(c)}$. Suppose $n \notin f(W_c)$. Then $W_c = F$, a contradiction, since $n \in f(F)$. So $n \in f(W_c)$. But then $W_c = W$, so $n \in f(W)$.

THEOREM. *Let f, g be recursive functions such that*

$$W_m = W_n \rightarrow (W_{f(m)} = W_{f(n)} \;\&\; W_{g(m)} = W_{g(n)} \;\&\; W_{f(g(m))} = W_{g(f(n))}).$$

Let $V = \bigcup_{k>0} (f \circ g)^k(\varnothing)$. Then V is r.e., $V = f(V) = g(V)$ and, for all V', if $V' = f(V') = g(V')$, then $V' \supseteq V$.

PROOF. Let $V_0 = \varnothing$, $V_{n+1} = f(g(V_n))$. Since $W_0 = \varnothing$, if we define $k(0) = 0$; $k(n+1) = f(g(k(n)))$, then $V_n = W_{k(n)}$ for all $n \geqq 0$. So

$$V = \bigcup_{n>0} V_n = \bigcup_{n>0} W_{k(n)}$$

is an r.e. set.

If h is well defined on the r.e. sets and W, W' are r.e. sets with $W \subseteq W'$, then, by the lemma,

$$h(W) = \bigcup h(F)(F \subseteq W) \subseteq \bigcup h(F)(F \subseteq W') = h(W').$$

We have $\varnothing \subseteq f(\varnothing)$, $\varnothing \subseteq g(\varnothing)$. So

$$g(\varnothing) \subseteq g(f(\varnothing)) = f(g(\varnothing)), \quad f(\varnothing) \subseteq f(g(\varnothing)).$$

Hence $V_0 \subseteq f(V_0) \subseteq V_1$, $V_0 \subseteq g(V_0) \subseteq V_1$. Inductively, assume $V_n \subseteq g(V_n) \subseteq V_{n+1}$, $V_n \subseteq f(V_n) \subseteq V_{n+1}$. Then

$$V_{n+1} = f(g(V_n)) \subseteq f(V_{n+1}),$$
$$V_{n+1} = g(f(V_n)) \subseteq g(V_{n+1}),$$

so that

$$g(V_{n+1}) \subseteq g(f(V_{n+1})) = f(g(V_{n+1})) = V_{n+2}$$

and

$$f(V_{n+1}) \subseteq f(g(V_{n+1})) = V_{n+2}.$$

Thus

$$V_{n+1} \subseteq f(V_{n+1}) \subseteq V_{n+2}, \quad V_{n+1} \subseteq g(V_{n+1}) \subseteq V_{n+2}.$$

So, for all $n \geq 0$, $V_n \subseteq f(V_n) \subseteq V_{n+1}$, $V_n \subseteq g(V_n) \subseteq V_{n+1}$.

This gives

$$V = \bigcup_{n>0} V_n = \bigcup_{n>0} f(V_n) = \bigcup_{n>0} g(V_n).$$

So

$$f(V) = \bigcup f(F)(F \subseteq V) = \bigcup_{n>0} (\bigcup f(F)(F \subseteq V_n)) = \bigcup_{n>0} f(V_n) = V,$$

and

$$g(V) = \bigcup g(F)(F \subseteq V) = \bigcup_{n>0} (\bigcup g(F)(F \subseteq V_n)) = \bigcup_{n>0} g(V_n) = V.$$

Also, V is the least common fixed point. For let V' be any other common fixed point. Trivially $V_0 = \varnothing \subseteq V'$; suppose $V_n \subseteq V'$. Then $g(V_n) \subseteq g(V') = V'$, so that

$$V_{n+1} = f(g(V_n)) \subseteq f(V') = V'.$$

Hence $V = \bigcup_{n>0} V_n \subseteq V'$.

The reader will detect a close connection between the above proof and Kleene's proof of his "first" recursion theorem [1, p. 66].

There is a version of Theorem 2 for partial recursive functions, rather than r.e. sets. One can replace W by ϕ, $=$ by $\simeq$, $\bigcup$ by "least common extension of" and $\supseteq$ by "extends."

We note the assumption of well-definedness in Theorem 2 is necessary. For let e_n be the Gödel number of the finite system of equations $h(\boldsymbol{n}) = \boldsymbol{n}$ where h is a function letter and $\boldsymbol{n}$ is the numeral for n. Then, for all n, $W_{e_n} = \{n\}$. Define

$$f(x) = \mu y(y \geqq x \ \& \ (\exists n) \ (y = e_n)),$$
$$g(x) = \mu y(y \geqq x \ \& \ (\exists n) \ (x \leqq e_n \ \& \ y = e_{n+1})).$$

Then $f(g(x)) = g(f(x)) = g(x)$, so f and g commute as number-theoretic functions. f and g are recursive, but for all x, $W_{f(x)} \neq W_{g(x)}$. Thus f and g cannot possibly have a common fixed point.

References

1. S. C. Kleene, *Introduction to metamathematics*, Van Nostrand, Princeton, N. J., 1952. MR **14**, 525.

2. Hartley Rogers, Jr., *Theory of recursive functions and effective computability*, McGraw-Hill, New York, 1967. MR **37** #61.

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