CONNECTIVITY OF THE GRAPHS OF SEMIRINGS: LIFTING AND PRODUCT

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1. **Introduction.** A *semiring* is a nonempty set R equipped with two binary operations, called addition $+$ and multiplication (denoted by juxtaposition), such that R is multiplicatively a semigroup and additively a commutative semigroup, and that the multiplication is distributive across the addition both from the left and from the right. Contrary to [1], the existence of a zero element in a semiring is not assumed.

By the *graph* $G(\mathcal{R})$ of a semiring R, we mean the nonoriented graph whose vertices consists of the set $\mathcal{R}$ of all proper subsemirings of R in which two vertices V_1 and V_2 are adjacent, or joined by an edge, if and only if $V_1 \cap V_2 \neq \square$, where $\square$ denotes the empty set. The graph $G(\mathcal{R})$ is said to be *connected* if and only if for each pair or vertices V and V' there exists a finite sequence, called a *path*, of vertices $V = V_1, V_2, \cdots, V_{n+1} = V'$ such that every two consecutive vertices are adjacent. It is conjectured [4] that the graph of a semiring with more than two elements is connected.[1] Similar problems have been considered by Bosák [2], Lin [3], Ponděliček [5], and the authors [4].

The purpose of this paper is to prove the following results. As an application of Theorem 1, we show that the graphs of free semirings are all connected.

THEOREM 1 (LIFTING). *If a semiring R has a homomorphic image R' whose graph is nonempty and connected, then the graph of R is connected.*

THEOREM 2 (PRODUCT). *If $\{R_\lambda \mid \lambda \in \Lambda\}$ is a family of semirings that contains a semiring whose graph is nonempty and connected, then the Cartesian product $\Pi_\lambda R_\lambda$ has a connected graph.*

Examples are given to show that the converses of Theorems 1 and 2 are both not true.

2. **Definitions and preliminaries.** If R_1 and R_2 are two semirings, we shall write $R_1 \times R_2$ for the semiring under the operations:

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[1] Example of semiring with two elements that has a disconnected graph exists. See Example 1.

$$(a, b) + (c, d) = (a + c, b + d),$$
$$(a, b)(c, d) = (ac, bd)$$

for all (a, b) and (c, d) in the Cartesian product $R_1 \times R_2$. Similarly, the Cartesian product $\Pi_\lambda R_\lambda$ of any family $\{R_\lambda\}$ of semirings is a semiring. A *homomorphism* from a semiring R to a semiring R' is a function $h: R \to R'$ such that $h(a+b) = h(a) + h(b)$ and $h(ab) = h(a)h(b)$ for all a and b in R. For any $x_1, x_2, \cdots, x_n$ in R, we shall write $\langle x_1, x_2, \cdots, x_n\rangle$ for the subsemiring of R generated by $\{x_1, x_2, \cdots, x_n\}$. Thus, $\langle x_1, x_2, \cdots, x_n\rangle$ is the intersection of all the subsemirings of R containing $\{x_1, x_2, \cdots, x_n\}$.

The trivial semiring, which consists of a single element alone, has an empty graph. We shall call a semiring *nontrivial* if it contains more than one element.

The only known example of a semiring with a disconnected graph is the following

EXAMPLE 1. Let R_2 be the set $\{0, 1\}$ together with the addition and the multiplication given by the following tables:

+	0	1
0	0	1
1	1	1

·	0	1
0	0	0
1	0	1

R_2 is a semiring with exactly two proper subsemirings $\{0\}$ and $\{1\}$ and hence its graph, which consists of two isolated vertices, is not connected.

3. Proofs of main results. For our later convenience, we first establish the following useful

LEMMA. *A semiring R is nontrivial if and only if there exists an element $x \in R$ such that $\langle x\rangle \neq R$.*

PROOF. Only the "only if" part needs a proof. We prove the contrapositive of this. Let R be a semiring such that $\langle y\rangle = R$ for every $y \in R$. For any $x \in R$, since $R = \langle x^2\rangle$ and since members of $\langle x^2\rangle$ are those polynomials in x^2 with natural numbers as coefficients (e.g. $2x^2 = x^2 + x^2$) and without any constant term, we have,

$$x = n_1x^{2k_1} + n_2x^{2k_2} + \cdots + n_px^{2k_p}$$

for some natural numbers $n_1, \cdots, n_p$; $k_1, \cdots, k_p$. Denote $e = n_jx^{2k_1-1} + n_2x^{2k_2-1} + \cdots + n_px^{2k_p-1}$. Then $x = ex$, which implies that

e is multiplicatively a left unit (=left identity) for $\langle x\rangle = R$. Since by assumption

$$R = \langle e\rangle = \{e, 2e, 3e, \cdots\}$$
$$= \langle 2e\rangle = \{2e, 4e, 6e, \cdots\}$$

we must have

$$e = 2me \quad \text{for some integer } m \geqq 1.$$

Let $z = (2m-1)e$. Then $e = z+e$, and $ze = (2m-1)ee = (2m-1)e = z$. Similarly $ez = z$. Thus, z is additively and multiplicatively the zero element for $\langle e\rangle = R$. Consequently, the semiring $\langle z\rangle = \{z\} = R$ is trivial.

COROLLARY. *A semiring has a nonempty graph if and only if it is nontrivial.*

PROOF. Straightforward.

PROOF FOR THEOREM 1. Let $h: R \to R'$ be a homomorphism of R onto R', and let V and W be any two proper subsemirings of R. We shall exhibit a path in $G(\mathfrak{R})$ connecting V and W: Since h is a homomorphism, $h(V)$ and $h(W)$ are two (not necessarily proper) subsemirings of R'. By virtue of the lemma we have just proved, if $h(V) = R' = h(W)$ then there exist $v \in V$ and $w \in W$ such that $\langle h(v)\rangle$ and $\langle h(w)\rangle$ are two proper subsemirings of R'. The last conclusion is obviously true when $h(V) \neq R'$ and/or $h(W) \neq R'$. The connectedness of the graph of R' implies that there is a path

$$\langle h(v)\rangle, V'_1, V'_2, \cdots, V'_n, \langle h(w)\rangle$$

in $G(\mathfrak{R}')$ connecting $\langle h(v)\rangle$ and $\langle h(w)\rangle$. It follows that

$$V, \overset{-1}{h}(\langle h(v)\rangle), \overset{-1}{h}(V'_1), \overset{-1}{h}(V'_2), \cdots, \overset{-1}{h}(V'_n), \overset{-1}{h}(\langle h(w)\rangle), W$$

forms a path in $G(\mathfrak{R})$ connecting V and W. Hence, the graph of R is connected.

PROOF FOR THEOREM 2. Let R_μ denote a member of the family $\{R_\lambda \mid \lambda \in \Lambda\}$ such that the graph of R_μ is nonempty and connected, and let

$$\pi_\mu : \prod_{\lambda\in\Lambda} R_\lambda \to R_\mu$$

be the μth projection. Then since π_μ is an epimorphism, by Theorem 1 the graph of $\Pi_{\lambda\in\Lambda} R_\lambda$ is connected.

DEFINITION. By a free *semiring* generated by a set S we mean a semiring $F \supset S$ such that for every function $g: S \to R$ from the set S to a semiring R, there exists a unique homomorphism h: $F \to R$ satisfying the commutativity of the following diagram:

$$\begin{tikzcd} S \arrow[r, phantom, "\subset"] \arrow[dr, "g"'] & F \arrow[d, "h"] \\ & R \end{tikzcd}$$

It is easy to see that for any given nonempty set S, a free semiring (generated by S) exists, and that all the free semirings generated by the same set are isomorphic to each other.

THEOREM 3. *The graph of a free semiring is connected.*

PROOF. Let F be a free semiring generated by a set S. Then S is necessarily nonempty. Choose any $x \in S$, and consider the subsemiring $\langle x \rangle$ of F generated by the single element x. Since $\langle x \rangle$ is a nontrivial commutative semiring, its graph is nonempty and connected [4]. Let g: $S \to \langle x \rangle$ be the function defined by $g(t) = x$ for all $t \in S$. Then since F is the free semiring generated by S, there exists a homomorphism h: $F \to \langle x \rangle$ such that $h | S = g$. It follows that $h(F) = \langle x \rangle$ and hence, by Theorem 1, the graph of F is connected.

4. **Remarks.** It is natural to ask whether the connectedness of the graphs of semirings is preserved by the epimorphisms, that is whether the converse of Theorem 1 is true. The following example shows that the answer is in the negative.

EXAMPLE 2. Let R_2 be the semiring of Example 1, and let $R = R_2 \times R_2$. A routine verification shows that the graph of R is connected. Let h: $R \to R_2$ be the first (or second) projection of R onto R_2. Then h is an epimorphism. However, the graph of $h(R) = R_2$ is, as we have seen in Example 1, not connected.

Example 2 also shows the converse of Theorem 2 is not true.

REFERENCES

1. Paul J. Allen, *A fundamental theorem of homomorphisms for semirings*, Proc. Amer. Math. Soc. **21** (1969), 412–416.

2. J. Bosák, *The graphs of semigroups*, Theory of graphs and its applications, Proc. Sympos. (Smolenice, 1963) Publ. House Czechoslovak Acad. Sci., Prague, 1964, pp. 119–125. MR **30** #3928.

3. Y.-F. Lin, *A problem of Bosák concerning the graphs of semigroups*, Proc. Amer. Math. Soc. **21** (1969), 343–346.

4. Y.-F. Lin and J. S. Ratti, *The graphs of semirings*, J. Algebra (to appear).

5. Bedřich Pondělíček, *Diameter of a graph of a semigroup*, Časopis Pěst. Mat. **92** (1967), 206–211. MR **36** #6323.

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