PROJECTING THE SPACE OF BOUNDED OPERATORS ONTO THE SPACE OF COMPACT OPERATORS

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This paper studies the existence of bounded projections from the space $B(X, Y)$ of bounded linear operators onto the subspace $K(X, Y)$ of compact linear operators, where X and Y are normed spaces.

For the cases where X and Y are either the l_p or c_0 spaces, the problem was first answered by E. Thorp [3]. For the cases where X is an infinite dimensional L-space or where Y is the space of all continuous functions on a topological space, various theorems were given by Arterburn and Whitley [1].

Owing to a recent result in [4], we will show that a more decisive answer can be given for the case where X, Y are Banach sequence spaces when the problem is considered from the approach of Thorp [3]. (See (1.2), (2.1), and (2.4).)

1. The definitions of terms and the theorem below are found in [4, Theorem 2.1].

(1.1) THEOREM. *Let λ be a monotone sequence space. Let μ be any sequence space. If $A:\lambda\to\mu$ is a matrix map and D is the associated diagonal submatrix, then $D(\lambda)\subset\mu^{\times\times}$.*

We call a sequence space λ monotone if it satisfies: $(c_1x_1, \cdots, c_ix_i, \cdots)\in\lambda$ whenever $(x_1, \cdots, x_i, \cdots)\in\lambda$ and $c_i=1$, -1 or 0 for all i. A monotone sequence space is said to be monotonely normed if, in addition,

$$\|(c_1x_1, \cdots, c_ix_i, \cdots)\| \leqq \|(x_1, \cdots, x_i, \cdots)\|.$$

(1.2) STANDING HYPOTHESES. Throughout this paper, we will assume that λ is monotonely normed and that μ is a perfect (normed) sequence space, i.e., $\mu^{\times\times}=\mu$ where $(\cdot)^{\times}$ denotes the Köthe dual. We will also assume that the unit ball V of μ satisfies $V^{\times\times}=V$ (if μ is Banach, this condition may be assumed without loss of generality). Consequently, μ will also be monotonely normed. All spaces of operators from λ to μ will be given the operator sup norm.

(1.3) THEOREM. *Let $M(\lambda, \mu)$ denote the space of bounded matrix maps from λ to μ and $DM(\lambda, \mu)$ the subspace of diagonal matrix maps. Then the map $\mathcal{P}$: $M(\lambda, \mu)\to DM(\lambda, \mu)$ which takes every matrix map*

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(a_{ij}) to its associated diagonal $(\delta_{ij}a_{ij})$ is a bounded projection of norm one.

PROOF. By (1.1), $\mathcal{P}$ is well defined. Let $A \in M(\lambda, \mu)$ and let D be the associated diagonal. If S is a set of indices, define $\pi_s(x_1, \cdots, x_i, \cdots)$ to have x_i as its ith coordinate if $i \in S$ and to have 0 as its ith coordinate if $i \notin S$. Let

$$S(n) = \{i: 1 \leqq i \leqq n\}.$$

Clearly $\|\pi_s\| \leqq 1$ when it acts on λ or μ. By Remark (1.5) of [4], we have:

$$\|\pi_{s(n)} A \pi_{s(n)}\| \geqq \|\pi_{s(n)} D \pi_{s(n)}\|.$$

Since μ is perfect,

$$\|D\| = \lim_n \|\pi_{s(n)} D \pi_{s(n)}\|$$

and since $\|A\| \geqq \|\pi_{s(n)} A \pi_{s(n)}\|$, we get $\|A\| \geqq \|D\| = \|\mathcal{P}(A)\|$. Thus $\mathcal{P}$ is a norm one projection.

2. (2.1) THEOREM. *Let $KM(\lambda, \mu)$ denote the space of all compact matrix maps from λ to μ. Let $KDM(\lambda, \mu)$ denote the space of all compact diagonal matrix maps. If there is a bounded projection from $M(\lambda, \mu)$ onto $KM(\lambda, \mu)$ then there is a bounded projection from $DM(\lambda, \mu)$ onto $KDM(\lambda, \mu)$.*

PROOF. Let $\iota: DM(\lambda, \mu) \to M(\lambda, \mu)$ be the inclusion map. Suppose $\mathcal{Q}: M(\lambda, \mu) \to KM(\lambda, \mu)$ is a bounded projection. If $\mathcal{P}$ is the projection in (1.3), then

$$\mathcal{P}\mathcal{Q}\iota: DM(\lambda, \mu) \to KDM(\lambda, \mu)$$

is a bounded projection.

(2.2) COROLLARY. *Suppose there is a bounded projection $\mathcal{H}$ mapping $K(\lambda, \mu)$ onto $KM(\lambda, \mu)$. If there is a bounded projection from $B(\lambda, \mu)$ onto $K(\lambda, \mu)$ then there is a bounded projection from $DM(\lambda, \mu)$ onto $KDM(\lambda, \mu)$.*

PROOF. Let $\mathcal{Q}_1: B(\lambda, \mu) \to K(\lambda, \mu)$ be a bounded projection. Then $\mathcal{H}\mathcal{Q}_1$ is a bounded projection from $M(\lambda, \mu)$ onto $KM(\lambda, \mu)$. Thus, (2.1) gives the bounded projection of $DM(\lambda, \mu)$ onto $KDM(\lambda, \mu)$.

(2.3) PROPOSITION. *Let λ be a Banach sequence space satisfying $\lambda \subset \lambda_0^{\times\times}$ where λ_0 is the closed linear span of all sequences $e_i = (0, 0, \cdots, 0, 1, 0, \cdots)$ where 1 occurs on the ith place. Then there is a bounded projection of λ' onto $\lambda^\times$ (when embedded in λ').*

PROOF. If $\iota: \lambda_0 \to \lambda$ is the inclusion map and R denotes the adjoint of ι, then R is a norm one projection of λ' onto $\lambda_0^\times$. Since $\lambda \subset (\lambda_0^\times)^\times$, we see that $\lambda_0^\times$ is contained in $\lambda^\times$. Thus $\lambda^\times = \lambda_0^\times$. They are isomorphic, by the open mapping theorem.

Combining (2.2) with (2.3), we get

(2.4) THEOREM. *If λ is a monotonely normed Banach sequence space satisfying $\lambda \subset \lambda_0^{\times\times}$, then the existence of a bounded projection from $B(\lambda, \mu)$ onto $K(\lambda, \mu)$ implies the existence of a bounded projection of $DM(\lambda, \mu)$ onto $KDM(\lambda, \mu)$.*

PROOF. It remains to show the existence of a bounded projection $\mathcal{K}: K(\lambda, \mu) \to KM(\lambda, \mu)$.

If T is a finite dimensional map from λ to μ given by:

$$T(x) = \sum_{1 \leq i \leq m} (x, \phi_i) y_i$$

where $\phi_i \in \lambda'$ and $y_i \in \mu$ and where (without loss of generality) $\|y_i\| = 1$ for all i, define $\mathcal{K}(T)$ to be the operator satisfying:

$$\mathcal{K}(T)(x) = \sum_{1 \leq i \leq m} (x, R\phi_i) y_i.$$

$\mathcal{K}: F(\lambda, \mu) \to FM(\lambda, \mu)$ defines a linear projection from the space of finite dimensional operators onto the subspace of finite dimensional matrix maps. To see that $\mathcal{K}$ is a norm one projection, we check that for any $\epsilon > 0$ and any $x \in \lambda$, there is a sufficiently large N for which:

$$|(x - \pi_{S(N)}(x), R\phi_i)| < \epsilon/m$$

for $i = 1, 2, \cdots, m$; but since

$$(\pi_{S(N)}(x), R\phi_i) = (\pi_{S(N)}(x), \phi_i),$$

we get

$$\begin{aligned}
\|T(\pi_{S(N)}(x))\| &= \Big\| \sum_{1 \leq i \leq m} (\pi_{S(N)}(x), \phi_i) y_i \Big\| \\
&= \Big\| \sum_{1 \leq i \leq m} (\pi_{S(N)}(x), R\phi_i) y_i \Big\| \\
&\geqq \Big\| \sum_{1 \leq i \leq m} (x, R\phi_i) y_i \Big\| - \Big\| \sum_{1 \leq i \leq m} (x - \pi_{S(N)}(x), R\phi_i) y_i \Big\| \\
&\geqq \|\mathcal{K}(T)(x)\| - \sum_{1 \leq i \leq m} |(x - \pi_{S(N)}(x), R\phi_i)| \, \|y_i\| \\
&\geqq \|\mathcal{K}(T)(x)\| - \epsilon.
\end{aligned}$$

Since x and ϵ are arbitrary and since $\|\pi_{S(N)}(x)\| \leqq \|x\|$, we get: $\|T\| \geqq \|\mathcal{K}(T)\|$. Thus $\mathcal{K}$ is a norm one projection and may therefore be continuously extended to map all of $K(\lambda, \mu)$ into the Banach subspace $KM(\lambda, \mu)$. The fixed points of $\mathcal{K}$ must contain the closure of $FM(\lambda, \mu)$, which is just $KM(\lambda, \mu)$, and so $\mathcal{K}$ is a norm one projection, as required.

(2.5) REMARKS. The following gives the typical application of (2.4). If $\lambda = l_p$ and $\mu = l_r$, where $1 \leqq p \leqq r \leqq \infty$, then all hypotheses of (2.4) are satisfied and it is easy to check that $DM(\lambda, \mu)$ is isometric with l_∞ while $KDM(\lambda, \mu)$ is isometric with c_0. Since there cannot be any bounded projection of l_∞ onto c_0 (see Phillips [2]), there cannot be any bounded projection of $B(\lambda, \mu)$ onto $K(\lambda, \mu)$.

Finally, we observe that (2.4) gives some new results when we apply it to the class of symmetric Köthe spaces, which, Garling [5] has shown, include spaces other than the l_p spaces.

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