COMMUTATIVE ALGEBRAS OF HOCHSCHILD DIMENSION ONE

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Let R be a commutative ring. Call a commutative R-algebra S *locally separable* if every finite set of elements of S is contained in a separable R-subalgebra which is finitely generated and projective as an R-module. If in addition the subring of S left fixed by the group $\mathrm{Aut}_R(S)$ of all R-algebra automorphisms of S is precisely R, S is called a *locally Galois* R-algebra. An element s of S is *almost separable* if for all g in $\mathrm{Aut}_R(S)$ $s-g(s)$ is either 0 or not a zero divisor.

THEOREM. *Let T be a locally Galois R-algebra with no idempotents except zero and one and let S be a locally separable R-subalgebra of T generated by almost separable elements. Then if S is of Hochschild dimension one, S is countably generated as an R-module.*

This theorem partially answers a question raised by Rosenberg and Zelinsky [5]. A similar result for the case of fields was obtained by MacRae [3]. We break the proof into lemmas, retaining throughout the notation of the hypotheses of the theorem. Unsubscripted tensors are over R. Thanks are due to the referee for some valuable suggestions.

Let $G=\mathrm{Aut}_R(T)$. G is the inverse limit of its restrictions to the separable R-subalgebras of T [4, Theorem 2, p. 338] and hence has a topology as a profinite group. Let T carry the discrete topology and let $C(G, T)$ be the ring of continuous T-valued functions on G. There is an R-algebra map $f: S\otimes S\to C(G, T)$ defined by $f(s\otimes t)(g)=sg(t)$.

LEMMA 1. *$f: S\otimes S\to C(G, T)$ is a monomorphism.*

PROOF. Let $x=\sum x_i\otimes y_i$ lie in the kernel of f and let V be a separable R-subalgebra of S, finitely generated and projective as an R-module, containing each x_i and y_i. Let W be the subalgebra of T generated by the conjugates of V under G; W is finitely generated and projective as an R-module and separable as an R-algebra, $\mathrm{Aut}_R(W)$ is finite, and the restriction map $G\to\mathrm{Aut}_R(W)$ is onto. (These facts are found in [4, Theorem 1, p. 335] and [1, Corollary 3.3, p. 25].) Let E be the ring of all W-valued functions on $\mathrm{Aut}_R(W)$ and let $h: W\otimes W\to E$ be defined by $h(w\otimes v)(g)=wg(v)$. By [1, Theorem 1.3,

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p. 18], h is an isomorphism. Moreover, the map of E to $C(G, T)$ induced by the restriction of G to $\mathrm{Aut}_R(W)$ and the inclusion of W into T is one-one. V, being separable over R, is a direct summand of W, so the composite $V\otimes V\to W\otimes W\to E\to C(G, T)$ is one-one. The image of $\sum x_i\otimes y_i$ (in $V\otimes V$) under this composite is the same as the image of x under f, so that $\sum x_i\otimes y_i=0$ in $V\otimes V$ and thus also in $S\otimes S$.

For y an element of $S\otimes S$, let $Z(y)$ be the set of all s in S such that $(s\otimes 1-1\otimes s)y=0$ and let $Z'(y)$ be the subring of S left fixed by the set U of all g in G such that $f(y)(g)\neq 0$.

LEMMA 2. *If y is nonzero, $Z'(y)$ is a finitely generated separable R-algebra. If s is an almost separable element of S, s is in $Z(y)$ if and only if s is in $Z'(y)$.*

PROOF. Let H be the subgroup of G generated by U and the group P of automorphisms fixing S. By [4, Theorem 3, p. 339], S is precisely the subring left fixed by P. Thus $Z'(y)$ is precisely the subring left fixed by H. Since y is not zero, Lemma 1 implies that U is a nonempty open subset of G. Thus H is open in G and $[G:H]$ is finite. By [4, Theorem 2, p. 338], then, $Z'(y)$ is a separable R-algebra, finitely generated as an R-module. The second assertion follows directly from Lemma 1.

PROOF OF THEOREM. To say that S has Hochschild dimension one means that the kernel J of the multiplication map of $S\otimes S$ to S is a projective $S\otimes S$-module. Let $\{f_i, x_i\}$ be a projective coordinate system for J. We may assume that no $f_i(x_i)$ is zero. For each x in J, only finitely many of the $f_i(x)$ are not zero, and since each f_i is $S\otimes S$-linear, $f_i(x)x_i=f_i(xx_i)=xf_i(x_i)$. For each i, let $S_i=Z(f_i(x_i))$ and let $T_i=Z'(f_i(x_i))$. Each element s of S lies in all but finitely many of the S_i, for $s\otimes 1-1\otimes s$ is in J and by the above only finitely many of the $(s\otimes 1-1\otimes s)f_i(x_i)$ are nonzero. Thus each almost separable element lies in all but finitely many T_i. There are infinitely many x_i: for if not, let $e=\sum f_i(x_i)$. For each s in S we have by the properties of a projective coordinate system $s\otimes 1-1\otimes s=\sum f_i(s\otimes 1-1\otimes s)x_i$ $=(s\otimes 1-1\otimes s)e$. Since e is clearly idempotent and S has no nontrivial idempotents, either e or $1-e$ lies in J. In the first case, the map of S to $S\otimes S$ sending s to $(s\otimes 1)(1-e)$ is an $S\otimes S$-inverse to the multiplication map of $S\otimes S$ to S and hence S is of dimension zero, contrary to hypothesis. The second case is similar. Since there are infinitely many x_i, each s in S lies in some S_i, and thus each almost separable s lies in some T_i. Since S is not separable, there are infinitely many T_i. Now let s be any element of S. There are almost separable elements $s_1, \cdots, s_n$ such that s is in $R[s_1, \cdots, s_n]$. Each s_k is in all but finitely many T_i, hence s is in all but finitely many T_i. It follows that any countable union of the T_i must be all of S. Since each T_i is finitely generated, S must be countably generated.

If s is an element of S such that $R[s]$ is a separable R-subalgebra of S (i.e. s is a separable element) then by [**2**, Lemma 2.1, p. 467, and Lemma 2.7, p. 469] s is almost separable. Thus the theorem applies when S is generated by separable elements. This will always happen if R is a local ring with infinite residue class field [**2**, Lemma 3.1, p. 471]. Also, if T is a domain clearly every element of S is almost separable.

Now if R is any connected (i.e. contains no nontrivial idempotents) commutative ring and if S is any connected, locally separable R-algebra, S can always be embedded in a connected locally Galois R-algebra, namely the separable closure of R [**2**, pp. 462–465]. If R is an integrally closed Noetherian domain, the separable closure of R is also a domain by [**2**, Corollary 4.2, p. 473]. Thus we have

COROLLARY. *Let R be an integrally closed Noetherian domain or a local ring with infinite residue class field. Then any connected, locally separable R-algebra of Hochschild dimension one is countably generated as an R-module*

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