# PRIMITIVE RINGS WITH INVOLUTION WHOSE SYMMETRIC ELEMENTS SATISFY A GENERALIZED POLYNOMIAL IDENTITY

WALLACE S. MARTINDALE III

Let $R$ be a primitive ring with involution $*$. Thus $R$ may be considered as an irreducible ring of endomorphisms of an additive abelian group $V$, so that $D = \mathrm{Hom}_R(V, V)$ is a division ring. Let $C$ be the center of $D$. We shall furthermore assume that $CR \subseteq R$. It can be shown that an involution $\gamma \to \bar{\gamma}$ is induced in $C$ which has the property that $\bar{\gamma}x = (\gamma x^*)^*$ for all $x \in R$. The involution $*$ is of the first kind if $\gamma \to \bar{\gamma}$ is the identity mapping and is of the second kind if there is a $\gamma \neq 0 \in C$ such that $\bar{\gamma} = -\gamma$. The set of symmetric elements of $R$ will be denoted by $S$.

We now assume that $S$ satisfies a nontrivial generalized polynomial identity over $C$ (in the sense of Amitsur). This means that there exists a nonzero element $f(x_1, x_2, \cdots, x_n)$ in the so-called $C$-universal product $R\langle x \rangle$ of the $C$-algebra $R$ and the free $C$-algebra $C[x_1, x_2, \cdots, x_n, \cdots]$ in noncommuting indeterminants $x_1, x_2, \cdots, x_n, \cdots$ such that $f(s_1, s_2, \cdots, s_n) = 0$ for all $s_1, s_2, \cdots, s_n \in S$. For more precise details concerning the above notions we refer the reader to [1, §4], and [2, §3]. The usual linearization process may be used so that we may assume without loss of generality that $S$ satisfies a nontrivial generalized (homogeneous) multilinear identity of degree $n$ in $x_1, x_2, \cdots, x_n$:

$$f = \sum \beta_k a_{i_0} x_{j_1} a_{i_1} \cdots a_{i_{n-1}} a_{j_n} a_{i_n} = 0 \tag{1}$$

where each monomial is of the same fixed degree $n$, $\beta_k \in C$, and the $a_k$'s are elements of $R$. Furthermore it is clear that one may assume that the $a_k$'s belong to some fixed $C$-basis of $R$, and that for two distinct monomials in which the variables appear in the same order the corresponding sequences $(a_{i_0}, a_{i_1}, \cdots, a_{i_n})$ and $(a_{i_0'}, a_{i_1'}, \cdots, a_{i_n'})$ of ring elements differ in at least one position.

Our object in this paper is to prove that, under the given conditions on our ring $R$, $D$ is finite dimensional over $C$ and $R$ contains nonzero transformations of finite rank. The proof rests heavily on an elementary but powerful lemma on vector spaces due to Amitsur [1, p. 211, Lemma 1], a specific version of which we state as follows:

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LEMMA 1 (AMITSUR). *Let $V$ be a vector space over a field $F$ and let $b_1, b_2, \cdots, b_m$ be $F$-independent linear transformations of $V$. Then for any finite dimensional subspace $U_0$ of $V$ either there exists $v \in V$ such that $vb_1, vb_2, \cdots, vb_m$ are independent modulo $U_0$ or there is a nonzero transformation $b = \sum_{i=1}^{m} \alpha_i b_i$, $\alpha_i \in F$, of finite rank.*

We are now ready to begin the study of the primitive ring $R$ with involution $*$, with $CR \subseteq R$, such that the symmetric elements $S$ satisfy a generalized multilinear identity of the form (1).

Without loss of generality we may assume that the involution $*$ is of the first kind. Indeed, if $\bar{\gamma} = -\gamma$ for some $\gamma \neq 0 \in C$, and $k \in K$, the set of skew elements of $R$, then $\gamma k \in S$ and consequently

$$f(\gamma k, s_2, \cdots, s_n) = \gamma f(k, s_2, \cdots, s_n) = 0$$

for all $s_2, \cdots, s_n \in S$, thus forcing $f(k, s_2, \cdots, s_n) = 0$. Repetition of this argument yields

$$f(s_1 + k_1, s_2 + k_2, \cdots, s_n + k_n) = 0$$

for all $s_i \in S$, $k_i \in K$. Because the characteristic of $R$ is unequal to two, every element of $R$ is of the form $s + k$, $s \in S$, $k \in K$, and so $R$ itself satisfies the same generalized multilinear identity. By a theorem of Amitsur [1, p. 218, Theorem 10], $[D:C] < \infty$ and $R$ contains nonzero transformations of finite rank.

Let $F$ be a maximal subfield of $D$. Following Amitsur [1, p. 215, Lemma 5] we note that $R_F$, the subring of Hom$(V, V)$ generated by $R$ and $F$, acts irreducibly on $V$ in the obvious way, with $\mathrm{Hom}_{R_F}(V, V) = F$. Since the involution $*$ of $R$ is of the first kind it may be extended to an involution (again denoted by $*$) of $R_F$ according to the rule

$$\left(\sum \alpha_i r_i\right)^* = \sum \alpha_i r_i^*, \qquad \alpha_i \in F, \quad r_i \in R.$$

[1, p. 215, Lemma 6(b)] insures that this mapping is well defined. Thus $FS$ is the set of symmetric elements of $R_F$ and satisfies the same generalized multilinear identity (1) as does $S$. Furthermore it is clear that (1) remains nontrivial over $F$.

At this point we claim that, in order to prove our main theorem, it suffices to show that $R_F$ contains a nonzero linear transformation of finite rank of $V$ over $F$. Indeed, this follows by [1, p. 216, Theorem 7]. Therefore, for the remainder of our proof, we are justified in assuming that to start with $D = C$ is a field. We suppose, for sake of argument, that $R$ does not contain a nonzero transformation of finite rank and aim at obtaining a contradiction.

LEMMA 2. *Let $v_1, v_2, \cdots, v_k$ be C-independent vectors in V, let $b_1, b_2, \cdots, b_m$ be C-independent elements of R, and let $U_0$ be a finite dimensional subspace of V. Then there exists $s \in S$ such that $v_1sb_1, v_1sb_2, \cdots, v_1sb_m$ are independent modulo $U_0$ and $v_is=0$, $i>1$.*

PROOF. First choose $x \in R$ such that $v_1x \neq 0$ and $v_ix=0$, $i>1$. Next note that $b_1^*, b_2^*, \cdots, b_m^*$ are $C$-independent in $R$. By Lemma 1, there exists $w \in V$ such that $wb_1^*, wb_2^*, \cdots, wb_m^*$ are independent modulo the subspace generated by $v_1, v_2, \cdots, v_k$. Thus one may pick $r \in R$ such that $v_ir=0$, $i=1, 2, \cdots, k$ and $wb_i^*r=wb_i^*$, $i=1, 2, \cdots, m$. If $\{b_i^*r\}$ is a dependent set, then $\sum \lambda_i(b_i^*r)=0$, some $\lambda_i \neq 0$. Consequently, $\sum \lambda_i(wb_i^*r) = \sum \lambda_i(wb_i^*)=0$, a contradiction to the independence of $\{wb_i^*\}$. Thus $\{b_i^*r\}$, and hence $\{r^*b_i\}$, is an independent set. Using Lemma 1 again, we can find $v \in V$ such that $\{vr^*b_i\}$ is independent modulo $U_0$. Now pick $t \in R$ so that $v_1xt=v$. Then $v_1(xtr^*+rt^*x^*)b_j=vr^*b_j$, $j=1, 2, \cdots, m$ and $v_i(xtr^*+rt^*x^*)=0$ for $i=2, 3, \cdots, k$. The proof of the lemma is complete, as we note $xtr^*+rt^*x^* \in S$.

Returning to the consideration of the generalized multilinear identity (1), we write (1) in the form $f=g+h$, where $g$ is the sum of all monomials in which the variables appear in the standard order (we may assume $g \neq 0$). We let $a_{01}, \cdots, a_{0k_0}$ be the distinct (and hence independent) elements of $R$ which appear before $x_1$ in the monomials comprising $g$, and in general we let $a_{i1}a_{i2}, \cdots, a_{ik_i}$ denote those distinct elements among the $C$-basis $\{a_i\}$ which appear between $x_i$ and $x_{i+1}$ in all those monomials belonging to $g$ which start out in the form $a_{01}x_1a_{11}x_2 \cdots a_{i-1,1}x_i \cdots$. We let $A$ denote the (necessarily finite dimensional) $C$-subspace of $R$ spanned by all the $a_k$'s appearing in (1).

By Lemma 1 we can choose $v \in V$ such that $va_{01}, va_{02}, \cdots, va_{0k_0}$ are independent. Let $W_0 = \sum_{j>1} Cva_{0j}$, and let $U_0=vA$. By Lemma 2 we pick $s_1 \in S$ such that $W_0s_1=0$ and $va_{01}s_1a_{11}, \cdots, va_{01}s_1a_{ik_1}$ are independent modulo $U_0$. Making repeated use of Lemma 2, we may choose a sequence

$$W_0, U_0, s_1, W_1, U_1, s_2, \cdots, W_{n-1}, U_{n-1}, s_n$$

as follows:

Let $W_i = \sum_{j>i} Cva_{01}s_1a_{11}s_2 \cdots a_{i-1,1}s_ia_{ij}$.

Let $U_i = U_0 + \sum vAs_{p_1}As_{p_2} \cdots As_{p_l}A$, where $p_j \leqq i$ and $l \leqq i$.

Choose $s_{i+1} \in S$ such that $\{va_{01}s_1a_{11} \cdots s_ia_{i1}s_{i+1}a_{i+1,j}\}$ is an independent set modulo $U_i$ and $W_is_{i+1}=0$ and $U_{i-1}s_{i+1}=0$.

Now substitute $s_1, s_2, \cdots, s_n$ in the identity (1). We claim that any monomial in which the variables are permuted becomes 0. Indeed, let $p$ be the first subscript not in the standard position, that is, the monomial has the form $b_0x_1b_1x_2 \cdots x_{p-1}b_{p-1}x_q \cdots$, with $q>p$. Thus $vb_0s_1b_1s_2 \cdots s_{p-1}b_{p-1} \in U_{p-1} \subseteq U_{q-2}$, and our claim is established since $U_{q-2}s_q=0$. Therefore $h(s_1, s_2, \cdots, s_n)=0$. On the other hand, by the way in which $s_1, s_2, \cdots, s_n$ were chosen, we finally see that

$$0 = f(s_1, s_2, \cdots, s_n) = \sum_j \lambda_j w_j$$

where not all $\lambda_j=0$ and $w_j=a_{01}s_1a_{11} \cdots a_{n-1,1}s_na_{nj}$. A contradiction results since $\{w_j\}$ is an independent set. This completes the proof of our main result, which we now state again.

THEOREM. *Let $R$ be a dense ring of linear transformations of a vector space $V$ over a division ring $D$, with $CR\subseteq R$, where $C$ is the center of $D$. Furthermore, assume that $R$ has an involution and that the set $S$ of symmetric elements of $R$ satisfies a generalized polynomial identity over $C$. Then $D$ is finite dimensional over $C$ and $R$ contains nonzero transformations of finite rank.*

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UNIVERSITY OF MASSACHUSETTS