ON NEARLY COMMUTATIVE NODAL ALGEBRAS IN CHARACTERISTIC ZERO

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ABSTRACT. In this paper we consider algebras satisfying the identity (I) $x(xy)+(yx)x=2(xy)x$ and show that there are no nodal algebras of this type over any field F of characteristic zero. The proof is obtained by first showing that if x is an element of a finite-dimensional algebra satisfying (I) over a field of characteristic zero then the operator $L(x)-R(x)$ is nilpotent.

A finite-dimensional power-associative algebra A with identity 1 over a field F is called a nodal algebra [7] if every x in A is of the form $x=\alpha 1+n$ where α is in F and n is nilpotent, and the set N of nilpotent elements of A does not form a subalgebra of A. Albert [1] has proved that there are no commutative nodal algebras over any field F of characteristic zero by showing that N forms a subalgebra. There do exist, however, examples of nodal algebras over fields of characteristic zero [2].

Algebras satisfying (I) have been studied by Kosier [5], Witthoft [8] and the author [6]. It should be noted that in linearized form (I) reduces to

$$x(zy)+z(xy)+(yx)z+(yz)x=2(xy)z+2(zy)x \tag{1}$$

and in operator form (I) is just

$$L(x)^2+R(x)^2=2L(x)R(x) \tag{2}$$

or

$$L(x)(L(x)-R(x))=(L(x)-R(x))R(x) \tag{3}$$

where $L(x)(R(x))$ is the operator which acts as follows: $yL(x)=xy(yR(x)=yx)$.

In a commutative algebra $L(x)-R(x)=0$. For algebras satisfying (I) we have the following.

THEOREM 1. *Let A be a finite-dimensional algebra satisfying* (I) *over a field F of characteristic zero. Then for any element x the operator $L(x)-R(x)$ is nilpotent.*

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PROOF. It is known [4, p. 43] that a transformation T on a finite-dimensional vector space V over a field of characteristic zero is nilpotent if all of its powers have trace zero; i.e., tr $T^n=0$ $(n=1, 2, \cdots)$. We show that $T=(L(x)-R(x))^2$ is nilpotent. Consider $\operatorname{tr}[(L(x)-R(x))^m]$ for $m\geqq 2$. Clearly,

$$\begin{aligned}\operatorname{tr}[(L(x)-R(x))^m] &= \operatorname{tr}[L(x)(L(x)-R(x))(L(x)-R(x))^{m-2}]\\ &\quad - \operatorname{tr}[R(x)(L(x)-R(x))^{m-2}(L(x)-R(x))]\end{aligned}$$

now, by (3), $L(x)(L(x)-R(x))=(L(x)-R(x))R(x)$ and by the commutativity of the trace, $\operatorname{tr}[R(x)(L(x)-R(x))^{m-2}(L(x)-R(x))]=\operatorname{tr}[(L(x)-R(x))R(x)(L(x)-R(x)^{m-2})]$. Therefore,

$$\begin{aligned}\operatorname{tr}[(L(x)-R(x))^m] &= \operatorname{tr}[(L(x)-R(x))R(x)(L(x)-R(x))^{m-2}]\\ &\quad - \operatorname{tr}[(L(x)-R(x))R(x)(L(x)-R(x))^{m-2}]\\ &= 0.\end{aligned}$$

Therefore, $(L(x)-R(x))^2$ is nilpotent.

COROLLARY. *For A as above,* $\operatorname{tr}L(x)=\operatorname{tr}R(x)$ *for any element x.*

It should be noted that if A is flexible then Theorem 1 is trivial since then (2) is just $(L(x)-R(x))^2=0$.

THEOREM 2. *There do not exist any nodal algebras satisfying $x(xy)+(yx)x=2(xy)x$ over any field F of characteristic zero.*

PROOF. Gerstenhaber [3, p. 29] has shown that in a commutative power-associative algebra over a field F of characteristic zero, the assumption that an element n is nilpotent implies that $R(n)$ is nilpotent. If an algebra A is not commutative this result can be applied to A^+ where A^+ is the same vector space as A but multiplication in A^+ is given by: $x\cdot y=\frac{1}{2}(xy+yx)$, xy the multiplication in A. Thus, if a is nilpotent in A then $R(a)^+=\frac{1}{2}(R(a)+L(a))$ is nilpotent and $\operatorname{tr}R(a)+\operatorname{tr}L(a)=0$.

Now let A be a finite-dimensional power-associative algebra satisfying (I) over a field F of characteristic zero, every element a of A being of the form $a=\alpha 1+n$ with n nilpotent. We show that the set N of nilpotent elements of A forms a subalgebra. As in [6, Theorem 1] write (1) in terms of operators to get,

$$R(y)L(x)+R(xy)+L(yx)+L(y)R(x)=2L(xy)+2R(y)R(x). \tag{4}$$

If we interchange x and y in (4) and subtract the result from (4) we have: $[L(y), R(x)]+[R(y), L(x)]+R([x, y])+L([y, x])=2L([x, y])+2[R(y), R(x)]$ (as usual $[A, B]$ denotes $AB-BA$) which gives rise to:

$$\text{tr}(R[x, y]) + \text{tr } L([y, x]) = 2 \text{ tr } L([x, y]). \tag{5}$$

Let $[x, y]=-[y, x]=\alpha 1+n$ with α in F and n nilpotent. By [3] tr $L(n)+$tr $R(n)=0$. By the corollary tr $L(n)=$tr $R(n)$. Therefore tr $L(n)=$tr $R(n)=0$. Thus (5) reduces to: tr $R(\alpha 1)-$tr $L(\alpha 1)=2$ tr $L(\alpha 1)$ or $2\alpha \dim A=0$. Therefore $\alpha=0$. In particular, if x and y are in N then $[x, y]$ is in N. But by [1] $xy+yx$ is in N. Therefore xy and yx are in N, N is a subalgebra and A is not a nodal algebra.

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