AN ELEMENTARY TAUBERIAN THEOREM OF NONLINEAR TYPE

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ABSTRACT. We find a nonlinear condition on $f \in C^2$ $(0, \infty)$ which, when coupled with the assumption lim inf $|f(t)|/t < \infty$, implies $f \sim At$ and $f' \to A$. A is some constant and $t \to \infty$ or $t \to 0+$.

In what follows the function $y(t)$ is of class $C'(0, \infty)$, ω is positive and measurable on $(-\infty, \infty)$, and ϕ is measurable on $(0, \infty)$. The motivation for the kind of theorems that follow is given by Saari [1], and Theorem B represents an improvement of his Theorems 1 and 2.

The results are true whether $t \to \infty$ or $t \to 0+$ provided they are read consistently throughout.

THEOREM A. *If*

(1) $$a \equiv \liminf y < \infty \quad \textit{and} \quad A \equiv \limsup y > -\infty$$

and

(2) $$y'(t) \leqq \omega(y(t))\phi(t),$$

where ϕ is integrable, then lim y *exists and is finite.*

PROOF. Choose two sequences t_n, τ_n, $t_n \geqq \tau_n$, both approaching ∞ [or both approaching 0] so that $Y_n \equiv y(t_n) \to A$, $y_n \equiv y(\tau_n) \to a$. By (2) and the positivity of ω

$$\int_{y_n}^{Y_n} \frac{dy}{\omega(y)} \leqq \int_{\tau_n}^{t_n} \phi(\tau)\, d\tau.$$

Let $n \to \infty$. Since ϕ is integrable, the right-hand side approaches zero.

Therefore $\int_a^A (dy/\omega(y)) \leqq 0$. Again because $\omega(y) > 0$ and $A \geqq a$ this is possible only if $A = a = \infty$; or $A = a = -\infty$; or $A = a$, finite. The first two possibilities are ruled out by (1). This completes the proof.

THEOREM B. *Let* $g \in C^2(0, \infty)$. *If*

(1′) $$\liminf |g(t)|/t < \infty$$

and

(2′) $$g''(t) \leqq \omega(g'(t))\phi(t),$$

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where ϕ is integrable, then the limits

$$\lim g'(t), \qquad \lim g(t)/t$$

exist and are equal.

PROOF. Let $y=g'(t)$. Then (1′) implies (1). For if $a=\infty$, then $g'\rightarrow\infty$, $g(t)/t\rightarrow\infty$, a violation of (1′); and similarly if $A=-\infty$. Moreover (2′) is just (2). So by Theorem A we can conclude that $A=\lim g'$ exists. It follows in turn that $\lim g(t)/t$ exists and equals A.

Theorem B improves Saari's theorems this way: in place of (1′) he requires $g(t)=O(t)$; in (2′) he requires $\phi\geqq 0$; and his conclusion is $g'(t)=O(1)$.

REFERENCE

1. D. G. Saari, *Some large O nonlinear Tauberian theorems*, Proc. Amer. Math. Soc. **21** (1969), 459–462.

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