THE HAUSDORFF SUMMABILITY OF FOURIER SERIES

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1. **Introduction.** Corresponding to a sequence $\{\mu_n\}$, the Hausdorff transformation of the sequence $\{s_n\}$ is

$$t_n = \sum_{\nu=0}^{n} \binom{n}{\nu} (\Delta^{n-\nu}\mu_\nu) s_\nu.$$

It is known (see, e.g., Hardy [1, Theorem 208]) that a necessary and sufficient condition for a Hausdorff transformation generated by a sequence $\{\mu_n\}$ to be regular is that there exists in the interval $[0, 1]$ a function $\chi(t)$ of bounded variation such that $\chi(0)=\chi(+0)=0$, $\chi(1)=1$, and $\mu_n=\int_0^1 x^n d\chi(x)$, $n=0, 1, 2, \cdots$.

Let $f(x)$ be a periodic function with period 2π, and integrable L in $(-\pi, \pi)$, and let

$$(1) \qquad \begin{aligned} f(x) &\sim \tfrac{1}{2}a_0 + \sum_{\nu=1}^{\infty} (a_\nu \cos \nu x + b_\nu \sin \nu x) \\ &= \sum_{\nu=0}^{\infty} A_\nu(x). \end{aligned}$$

We write

$$\phi(t) = \phi_x(t) = f(x+t) + f(x-t) - 2s,$$

$$h(t) = h_x(t) = \int_0^t \phi(u)du,$$

$$H(t) = H_x(t) = \int_0^t |\phi(u)| du,$$

$$\overset{+}{g_\delta}(x) = \frac{1}{\Gamma(\delta)} \int_0^x (x-v)^{\delta-1} g(v)dv, \qquad (\delta > 0),$$

$$\overset{-}{g_\delta}(x) = \frac{1}{\Gamma(\delta)} \int_x^1 (v-x)^{\delta-1} g(v)dv, \qquad (\delta > 0).$$

We shall prove

THEOREM 1. *If $h(x)=o(t)$, $H(t)=O(t)$, then $\sum_{\nu=0}^{\infty} A_\nu(x)$ is summable Hausdorff, with*

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$$\chi(x) = \overset{+}{g}_{1+\delta}(x) + c, \qquad (\delta > 0) \quad \text{or} \tag{2}$$
$$= \overset{-}{g}_{1+\delta}(x) + c, \qquad (\delta > 0),$$

where $g(x)$ *is Lebesgue integrable over* $[0, 1]$, *to* s.

This theorem includes as a special case the following theorem, which is a slight generalization of a well-known result of Lebesgue and M. Riesz (see, e.g., Zygmund [**3**, p. 94]).

THEOREM 2. *If* $h(t) = o(t)$, $H(t) = O(t)$, *then* $\sum_{\nu=0}^{\infty} A_\nu(x)$ *is summable* (C, α) $(\alpha > 0)$ *to* s.

If $\chi(x)$ is the characteristic function of the closed interval $\alpha \leqq t \leqq 1$ $(0 < \alpha < 1)$, then $\mu_\nu = \alpha^\nu$, and the Hausdorff transformation reduces to the Euler transformation

$$t_n = \sum_{\nu=0}^{n} \binom{n}{\nu} \alpha^\nu (1 - \alpha)^{n-\nu} s_\nu.$$

It is known that the Euler summability method is not Fourier effective (see, e.g., Hardy [**1**, p. 360 and p. 364]). However we have

THEOREM 3. *If* $H(t) = o(t(\log t^{-1})^{-1})$, *then* $\sum_{\nu=0}^{\infty} A_\nu(x)$ *is summable Euler to* s.

The conjugate series of the Fourier series (1) is

$$\sum_{\nu=1}^{\infty} (a_\nu \sin \nu x - b_\nu \cos \nu x). \tag{3}$$

We write

$$\psi(t) = \psi_x(t) = f(x + t) - f(x - t),$$
$$G(t) = G_x(t) = \int_0^t |\psi(u)| \, du,$$
$$\bar{f}(x, h) = -\frac{1}{\pi} \int_h^\pi \frac{\psi(u)}{2 \tan(u/2)} \, du.$$

We have the following generalization of a result in Cesàro summability (see Zygmund [**3**, Theorem (5.8) p. 95]).

THEOREM 4. *If* $G(t) = o(t)$, *then*

$$\bar{h}_n(x) - \bar{f}(x, a/n) = o(1) \qquad (a > 0),$$

where $\bar{h}_n(x)$ *is the Hausdorff transformation, with* $\chi(x)$ *defined by* (2), *of the sequence* $\{\bar{s}_n(x)\}$ *of the partial sums of the series* (3).

We also have

THEOREM 5. *If* $G(t)=o(t(\log t^{-1})^{-1})$, *then*

$$\overline{E}_n(x) - \bar{f}(x, a/n) = o(1) \qquad (a > 0),$$

where $\overline{E}_n(x)$ *is the Euler transformation of* $\{\bar{s}_n(x)\}$.

2. **Proof of Theorem** 1. Let $h_n(x)$ be the Hausdorff transformation of the sequence $\{s_n\}$ of partial sums of the Fourier series (1). Then, since

$$s_n - s = \frac{1}{\pi}\int_0^\pi \phi(u)\frac{\sin nu}{u}\,du + o(1),$$

(4)
$$\begin{aligned} h_n(x) - s &= \frac{1}{\pi}\int_0^\pi \phi(u)K_n(u)du + o(1) \\ &= \frac{1}{\pi}\left(\int_0^{1/n} + \int_{1/n}^{A/n} + \int_{A/n}^{\pi}\right) + o(1) \\ &= (I_n' + I_n'' + I_n''')/\pi + o(1), \end{aligned}$$

where $A>1$ is a (temporarily) fixed constant, $n>A/\pi$, and

$$K_n(u) = \sum_{\nu=0}^{n}\binom{n}{\nu}(\Delta^{n-\nu}\mu_\nu)\frac{\sin \nu u}{u}.$$

When $0<\lambda u\leqq 1$, $\sin \lambda u/u$ is a monotonic decreasing function of u. Hence, by the mean-value theorem, there exists a δ_n $(0\leqq\delta_n\leqq 1)$ such that

(5)
$$\begin{aligned} I_n' &= \left(\sum_{\nu=0}^{n}\nu\binom{n}{\nu}(\Delta^{n-\nu}\mu_\nu)\right)\int_0^{\delta_n/n}\phi(u)du \\ &= o\left(\frac{1}{n}\int_0^1\left(\sum_{\nu=0}^{n}\nu\binom{n}{\nu}x^\nu(1-x)^{n-\nu}\right)\left|d\chi(x)\right|\right) \\ &= o\left(\int_0^1 x\left|d\chi(x)\right|\right) \qquad \text{(by [2, Lemma 3])} \\ &= o(1). \end{aligned}$$

We have

$$
\begin{aligned}
I_n'' &= [h(u)K_n(u)]_{1/n}^{A/n} - \int_{1/n}^{A/n} h(u)\frac{d}{du}K_n(u)du \\
&= \left[o(u)O\left(\frac{1}{u}\right)\right]_{1/n}^{A/n} + \int_{1/n}^{A/n} o(u)O\left(\frac{n}{u}\right)du \qquad (6)\\
&= o(1).
\end{aligned}
$$

Without loss of generality we may assume $0<\delta<1$. If $\chi(x)=g_{1+\delta}^{+}(x)+c$, then

$$
\begin{aligned}
K_n(u) &= \frac{1}{u}\sum_{\nu=0}^{n}\binom{n}{\nu}\sin \nu u\int_0^1 x^\nu(1-x)^{n-\nu}d\chi(x) \\
&= \frac{1}{u}\int_0^1\left(\sum_{\nu=0}^{n}\binom{n}{\nu}x^\nu(1-x)^{n-\nu}\sin\nu u\right)d\chi(x) \\
&= \frac{1}{u}\int_0^1 N_n(x,u)d\chi(x) \\
&= \frac{1}{u}\int_0^1 N_n(x,u)g_\delta^{+}(x)dx \\
&= \frac{1}{u}\int_0^1 N_{n\delta}^{-}(x,u)g(x)dx,
\end{aligned}
$$

where

$$
N_{n\delta}^{-}(x,u) = \frac{1}{\Gamma(\delta)}\int_x^1 (v-x)^{\delta-1}N_n(v,u)dv = O\left(\frac{1}{(nu)^\delta}\right)
$$

uniformly in $0\leqq x\leqq 1$ (by [2, p. 22, last six lines]). It follows that

$$
K_n(u) = O(1/n^\delta u^{1+\delta}).
$$

Hence

$$
\begin{aligned}
I_n''' &= O\left(\frac{1}{n^\delta}\int_{A/n}^{\pi}\frac{|\phi(u)|}{u^{1+\delta}}du\right) \\
&= O\left\{\frac{1}{n^\delta}\left[\frac{H(u)}{u^{1+\delta}}\right]_{A/n}^{\pi} + \frac{1+\delta}{n^\delta}\int_{A/n}^{\pi}\frac{H(u)}{u^{2+\delta}}du\right\} \qquad (7)\\
&= O(A^{-\delta}) + o(1).
\end{aligned}
$$

From (4), (5), (6) and (7), we get

$$
h_n(x) - s = O(A^{-\delta}) + o(1).
$$

Hence $\lim_{n\to\infty} \sup|h_n(x)-s| = O(A^{-\delta})$. Since A is arbitrarily large, $\lim_{n\to\infty} h_n(x) = s$.

The other case may be proved in a similar way.

3. **Proof of Theorem** 3. Let $E_n(x)$ be the Euler transformation of $\{s_n(x)\}$. Then

$$\begin{aligned} E_n(x) - f(x) &= \frac{1}{\pi}\int_0^\pi \frac{\phi(u)}{u}\left[\sum_{\nu=1}^n \binom{n}{\nu}\alpha^\nu(1-\alpha)^{n-\nu}\sin \nu u\right]du + o(1) \\ &= \frac{1}{\pi}\left(\int_0^{\log n/n} + \int_{\log n/n}^{\log n/n^{1/2}} + \int_{\log n/n^{1/2}}^{\pi}\right) + o(1) \qquad (8) \\ &= (I_n' + I_n'' + I_n''')/\pi + o(1). \end{aligned}$$

It follows from $\sin \nu u \leqq \nu u$ and

$$\sum_{\nu=1}^n \nu\binom{n}{\nu}\alpha^\nu(1-\alpha)^{n-\nu} = O(n)$$

that

$$I_n' = O\left(n\int_0^{\log n/n} |\phi(u)|\,du\right) = o(1). \tag{9}$$

We have

$$\begin{aligned} I_n'' &= O\left(\int_{\log n/n}^{\log n/n^{1/2}} \frac{|\phi(u)|}{u}\,du\right) \\ &= O\left\{\left[\frac{H(u)}{u}\right]_{\log n/n}^{\log n/n^{1/2}} + \int_{\log n/n}^{\log n/n^{1/2}} \frac{H(u)}{u^2}\,du\right\} \qquad (10) \\ &= O\left(-\int_{\log n/n}^{\log n/n^{1/2}} \frac{du}{u\log u}\right) + o(1) \\ &= o(1). \end{aligned}$$

Let (see [2, p. 23])

$$\rho(u) = [1 - 4\alpha(1-\alpha)\sin^2(u/2)]^{1/2},$$
$$\theta(u) = \tan^{-1}(\alpha\sin u/(1-\alpha+\alpha\cos u)).$$

Then, since $\rho(u) \leqq e^{-cu^2}$, where c is a positive constant,

$$\begin{aligned} I_n''' &= \int_{\log n/n^{1/2}}^{\pi} \frac{\phi(u)}{u}(\rho(u))^n \sin n\theta(u)\,du \\ &= O\left(\frac{n^{1/2}\exp[-c\log^2 n]}{\log n}\int_{\log n/n^{1/2}}^{\xi_n} |\phi(u)|\,du\right) \qquad (11) \\ &= o(1), \end{aligned}$$

where $\log n/n^{1/2} \leqq \xi_n \leqq \pi$. The theorem follows from (8), (9), (10) and (11).

4. **Proof of Theorem 4.** We have

$$\bar{s}_\nu(x) = -\frac{1}{\pi}\int_0^\pi \frac{\psi(u)}{\tan(u/2)}\sin^2\frac{\nu u}{2}\,du + o(1).$$

Hence

$$\begin{aligned}
\bar{h}_n(x) - \bar{f}\left(x, \frac{a}{n}\right)
&= -\frac{1}{\pi}\int_0^{a/n} \frac{\psi(u)}{\tan(u/2)}\left(\sum_{\nu=0}^{n}\binom{n}{\nu}(\Delta^{n-\nu}\mu_\nu)\sin^2\frac{\nu u}{2}\right)du \\
&\quad + \frac{1}{\pi}\int_{a/n}^{\pi} \frac{\psi(u)}{2\tan(u/2)}\left(\sum_{\nu=0}^{n}\binom{n}{\nu}(\Delta^{n-\nu}\mu_\nu)\cos \nu u\right)du + o(1) \\
&= O\left(n^2\int_0^{a/n} u\,|\psi(u)|\,du\right) + \frac{1}{\pi}\int_{a/n}^{\pi}\frac{\psi(u)}{2\tan(u/2)}L_n(u)du + o(1) \\
&= \frac{1}{\pi}\int_{a/n}^{\pi}\frac{\psi(u)}{2\tan(u/2)}L_n(u)du + o(1).
\end{aligned}$$

We first consider the case in which $\chi(x) = g^{+}_{1+\delta}(x) + c$. Then (see [2, Lemma 5 and p. 22, last six lines, with Im replaced by Re])

$$\begin{aligned}
L_n(u) &= \int_0^1 \left(\sum_{\nu=0}^{n}\binom{n}{\nu}x^\nu(1-x)^{n-\nu}\cos \nu u\right)d\chi(x) \\
&= \int_0^1 M_n(x, u)d\chi(x) \\
&= \int_0^1 M_n(x, u)g^{+}_{\delta}(x)dx \\
&= \int_0^1 M^{-}_{n\delta}(x, u)g(x)dx \\
&= \int_0^1 O((nu)^{-\delta})g(x)dx \\
&= O((nu)^{-\delta}).
\end{aligned}$$

Hence

$$\bar{h}_n(x) - \bar{f}\left(x, \frac{a}{n}\right) = O\left(n^{-\delta}\int_{a/n}^{\pi}\frac{|\psi(u)|}{u^{1+\delta}}du\right) = o(1).$$

The other case may be proved in a similar way.

5. **Proof of Theorem** 5. We have

$$\begin{aligned}
&\bar{E}_n(x) - \bar{f}\left(x, \frac{a}{n}\right) \\
&= -\frac{1}{\pi}\int_0^{a/n} \frac{\psi(u)}{\tan(u/2)}\left(\sum_{\nu=0}^{n}\binom{n}{\nu}\alpha^\nu(1-\alpha)^{n-\nu}\sin^2\frac{\nu u}{2}\right)du \\
&\quad + \frac{1}{\pi}\int_{a/n}^{\pi} \frac{\psi(u)}{2\tan(u/2)}\left(\sum_{\nu=0}^{n}\binom{n}{\nu}\alpha^\nu(1-\alpha)^{n-\nu}\cos \nu u\right)du + o(1) \\
&= O\left(n^2\int_0^{a/n} u\,|\psi(u)|\,du\right) + O\left(\int_{a/n}^{\log n/n^{1/2}} \frac{|\psi(u)|}{u}\,du\right) \\
&\quad + O\left(\frac{n^{1/2}\exp[-c\log^2 n]}{\log n}\int_{\log n/n^{1/2}}^{\xi_n'} |\psi(u)|\,du\right) + o(1) \\
&= o(1).
\end{aligned}$$

References

1. G. H. Hardy, *Divergent series*, Clarendon Press, Oxford, 1949. MR **11**, 25.

2. N. Tripathy, *On the absolute Hausdorff summability of Fourier series*, J. London Math. Soc. **44** (1969), 15–25. MR **38** #2523.

3. A. Zygmund, *Trigonometric series*. Vol. 1, 2nd rev. ed., Cambridge Univ. Press, New York, 1959. MR **21** #6498.

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