A NONCONTRACTIVE FIXED POINT THEOREM[1]

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Let B denote a Banach space with norm $\|\cdot\|$ and Y the Banach space of all continuous functions y on $[0, 1]$ to B with norm $|\cdot|$ defined by $|y| = \sup_{t\in(0,1)}\|y(t)\|$. Further, let X be a subset of Y for which it is true that $x, y \in X$ and $x(t_0) = y(0)$ implies u defined by $u(t) = x(t)$ for $t \in [0, t_0]$ and $u(t) = y(t-t_0)$ for $t \in [t_0, 1]$ is in X.

THEOREM. *If T is a continuous mapping of X into X ($x \to Tx$) the range of which is compact and for which*

(i) *$\exists$ a constant function $u \in X \ni (Tu)(0) = u(0)$ and*

(ii) *$x, y \in X$ and $x(t) = y(t)$ $\forall t \in [0, t_0]$, $0 < t_0 \leqq 1$, implies $Tx(t) = Ty(t)$ $\forall t \in [0, t_0]$, then $\exists x \in X \ni Tx = x$ and $x(0) = u(0)$.*

The proof is an adaptation of an existence proof for ordinary differential equations due to Carathéodory [1] (see also [2]). Note that it does not require that the domain of T be convex as does the Schauder-Tychonoff fixed point theorem [4].

PROOF. Let n denote any positive integer $\geqq 2$. Set $x_{n1} = u$ and then define for $j = 2, 3, \cdots, n$, in succession,

$$\begin{aligned} x_{nj}(t) &= u(0), & t &\in [0, 1/n], \\ &= Tx_{nj-1}(t - 1/n), & t &\in [1/n, 1]. \end{aligned}$$

Note that $x_{nj} \in X$ for each j. Note further that $x_{n2}(t) = x_{n1}(t), t \in [0, 1/n]$, and hence by induction that $x_{nn}(t) = x_{nn-1}(t), t \in [0, (n-1)/n]$. From this then, $x_{nn} = x_n$ is such that

$$\begin{aligned} x_n(t) &= u(0), & t &\in [0, 1/n], \\ &= Tx_n(t - 1/n), & t &\in [1/n, 1]. \end{aligned} \tag{1}$$

The sequence Tx_n so constructed contains a subsequence Tx_{n_k} which converges to some $x \in X$ and which, by Ascoli's theorem [3], is equicontinuous. It follows in turn from (1) that the sequence x_{n_k} is equicontinuous. To complete the proof it suffices to show $x_{n_k}(t) \to x(t)$ in B. For then the sequence x_{n_k} is compact, again by Ascoli's theorem, so that $x_{n_k} \to x$; hence $Tx_{n_k} \to Tx$, which together with $Tx_{n_k} \to x$ yields $Tx = x$. That indeed $x_{n_k}(t) \to x(t)$ is immediate from the fact that $x_{n_k}(0) = u(0) = x(0)$ and that, for $t \neq 0$ and sufficiently large k,

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$$\begin{aligned}\|x_{n_k}(t) - x(t)\| &= \|Tx_{n_k}(t - 1/n_k) - x(t)\| \\ &\leqq \|Tx_{n_k}(t - 1/n_k) - Tx_{n_k}(t)\| + \|Tx_{n_k}(t) - x(t)\|.\end{aligned}$$

References

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3. J. Dieudonné, *Foundations of modern analysis*, Pure and Appl. Math., vol. 10, Academic Press, New York, 1960. MR **22** #11074.

4. R. E. Edwards, *Functional analysis. Theory and applications*, Holt, Rinehart and Winston, New York, 1965. MR **36** #4308.

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