ON FINITE-DIMENSIONAL TORSION-FREE MODULES AND RINGS

E. P. ARMENDARIZ

1. This note is concerned primarily with a study of modules having zero singular submodule, called torsion-free modules, over finite-dimensional rings. In some of our results we do not require that the ring be finite-dimensional but only that direct sums of torsion-free injective modules be injective, a property shown by Mark Teply to be equivalent to the ring containing no infinite direct sum of torsion-free left ideals. As might be expected, torsion-free modules over these latter rings behave very much like the usual torsion-free modules over (commutative) integral domains. For example, we show in Theorem 1 that direct sums of torsion-free injectives are injective if and only if every torsion-free module contains a unique maximal injective submodule. Also, in analogy to the case for Abelian groups, we establish that a torsion-free module over a finite-dimensional torsion-free ring contains a homomorphic image of every nonzero torsion-free module if and only if it contains a faithful injective submodule (Theorem 3). This enables us to show in Theorem 4 that a semiprime finite-dimensional torsion-free ring has a projective injective envelope if and only if it is semisimple Artinian. A result for torsion-free prime rings without any restricted chain conditions similar to Theorem 3 is also obtained. Finally, over self-injective finite-dimensional rings we show that all torsion-free modules are injective and completely reducible (Corollary 6).

We assume throughout that R is a ring with identity and all R-modules are unitary left R-modules.

2. The singular submodule of an R-module M will be denoted by $Z(M)$. Following A. W. Goldie [6], M is *torsion-free* if $Z(M)=0$ and R is a torsion-free ring if R is a torsion-free R-module. An R-module M is *finite-dimensional* if it contains no infinite direct sum of submodules. For an R-module M, $E(M)$ will denote the injective envelope of M.

The following basic lemma is our starting point.

LEMMA 1. *Let Q be an injective R-module and M a torsion-free R-module. Then for any $\alpha \in \mathrm{Hom}_R(Q, M)$,* Ker α *is a direct summand of Q and hence* Im α *is an injective R-module.*

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PROOF. If $\alpha \in \mathrm{Hom}_R(Q, M)$ then Im α is torsion-free since M is torsion-free. If Ker α is an essential submodule of a submodule $P \subseteq Q$, then $Z(P/\mathrm{Ker}\ \alpha) = P/\mathrm{Ker}\ \alpha$ and it follows that $P/\mathrm{Ker}\ \alpha = 0$, being isomorphic to a torsion-free module. Thus Ker α has no essential extension in Q, hence Ker α is injective and so a direct summand of Q.

The next lemma is well known and easily established.

LEMMA 2. *A direct sum or direct product of torsion-free R-modules is torsion-free.*

If a ring R is left Noetherian then any direct sum of injective R-modules is injective [3, p. 17]. In [9, Theorem 2.1] Mark Teply has given several conditions on a ring R which are equivalent to the following property (see also [8]):

(*) Any direct sum of torsion-free injective R-modules is injective.

One of these, which we will use later, is that R contain no infinite direct sum of torsion-free left ideals. Thus any finite-dimensional ring R has property (*). Our first result gives another condition equivalent to (*) in terms of the injective submodules of torsion-free R-modules.

THEOREM 1. *A necessary and sufficient condition for a ring R to have property* (*) *is that every torsion-free R-module has a* (*unique*) *maximal injective submodule.*

PROOF. Suppose first that R has property (*). Let M be a torsion-free R-module, $\{H_\lambda \mid \lambda \in \Lambda\}$ the set of injective submodules of M, and $D = \sum_{\lambda \in \Lambda} H_\lambda$. If $H = \oplus \sum_{\lambda \in \Lambda} H_\lambda$, then by (*) H is injective since each H_λ is torsion-free, and D is a homomorphic image of H. Since D is torsion-free, D is injective by Lemma 1. Evidently, D is the maximal injective submodule of M. For the sufficiency let $Q = \oplus \sum_{\lambda \in \Lambda} Q_\lambda$, where each Q_λ is torsion-free and injective. Then Q is torsion-free by Lemma 2, so Q has a maximal injective submodule $D \neq 0$. If A is any injective submodule of Q, then $D + A$ is a torsion-free image of the injective module $D \oplus A$ hence is injective by Lemma 1. By maximality of D, $A \subseteq D$. In particular $Q_\lambda \subseteq D$ for all $\lambda \in \Lambda$ and thus $Q = D$ is injective.

We now give an application of Theorem 1. We note first that since a torsion-free self-injective ring is regular, it is semisimple Artinian if and only if it is finite-dimensional.

THEOREM 2. *If R is a semiprime finite-dimensional torsion-free ring then* $R = K \oplus S$ (*ring direct sum*), *where K is a semisimple Artinian ring and S is a semiprime finite-dimensional torsion-free ring containing no nonzero S-injective left ideals.*

PROOF. By Theorem 1, R contains a maximal injective left ideal K, hence $R=K\oplus S$. If $r\in R$ then Kr is a homomorphic image of K under right multiplication by r and so Kr is an injective R-module by Lemma 1. Thus $Kr\subseteq K$ and so K is a two-sided ideal of R. Now $KS\subseteq K\cap S=0$ and so $(SK)^2=0$. Since R contains no nonzero nilpotent ideals, $SK=0$ and this implies S is also a two-sided ideal of R. The remaining assertions concerning S are easily established.

If R is a left Noetherian ring then $Z(R)$ is a nilpotent ideal of R, [5], so if in addition R is semiprime then $Z(R)=0$. Thus we have

COROLLARY 1. *If R is a left Noetherian semiprime ring then $R=K\oplus S$ (ring direct sum), where K is semisimple Artinian and S is a left Noetherian semiprime ring containing no nonzero S-injective left ideals.*

COROLLARY 2. *A finite-dimensional ring R is semisimple Artinian if and only if every nonzero left ideal contains a nonzero injective left ideal.*

PROOF. Since $Z(R)$ can contain no nonzero idempotents we have $Z(R)=0$. The same holds for any nilpotent left ideal. Thus R is also semiprime.

COROLLARY 3. *A finite-dimensional torsion-free prime ring R is simple Artinian if and only if R contains a nonzero injective left ideal.*

If R is a torsion-free ring then for an R-module M, $Z(M)=M$ if and only if $\mathrm{Hom}_R(M, E(R))=0$ [4, Proposition 1]. Thus the torsion-free R-modules are precisely those R-modules which are embeddable in a direct product of copies of $E(R)$. Thus $E(R)$ can be considered a *cogenerator* for the class of torsion-free R-modules. We will use Theorem 1 to characterize all torsion-free cogenerators for the torsion-free R-modules whenever R is a finite-dimensional torsion-free ring. Note that a torsion-free R-module $M\neq 0$ is a cogenerator for the torsion-free R-modules if and only if $\mathrm{Hom}_R(B, M)\neq 0$ for all torsion-free R-modules $B\neq 0$. Recall that a left R-module M is *faithful* if $(0:M)=\{x\in R \mid xM=0\}=\{0\}$.

THEOREM 3. *Let R be a finite-dimensional torsion-free ring and let $M\neq 0$ be a torsion-free R-module. Then* $\mathrm{Hom}_R(B, M)\neq 0$ *for all torsion-free R-modules B if and only if M contains a nonzero faithful injective submodule.*

PROOF. Assume first that $\mathrm{Hom}_R(B, M)\neq 0$ for all torsion-free R-modules $B\neq 0$. Since R is torsion-free, $E(R)$ is torsion-free and so $\mathrm{Hom}_R(E(R), M)\neq 0$. Thus M contains a nonzero injective submodule by Lemma 1. Hence the maximal injective submodule D of M, which exists by Theorem 1, is nonzero. Let $K = \bigcap \{\operatorname{Ker} \alpha \mid \alpha \in \operatorname{Hom}_R (E(R), D)\}$. Then $E(R)/K$ can be embedded in a direct product of copies of D. By Lemma 2, $E(R)/K$ is a torsion-free R-module. Thus K can have no essential extension in $E(R)$ and so K is injective. For any $\alpha \in \operatorname{Hom}_R(K, M)$, $\alpha(K)$ is injective by Lemma 1, and so $\alpha(K) \subseteq D$. Thus α extends to $\beta \in \operatorname{Hom}_R (E(R), D)$ and $K \subseteq \operatorname{Ker} \beta$. Hence $\alpha(K) = 0$ and so $\operatorname{Hom}_R (K, M) = 0$. It follows that $K = 0$ and thus $E(R)$ is contained in a direct product of copies of D. Since any element in $(0:D)$ annihilates any direct product of copies of D and so also $R \subseteq E(R)$, we conclude that $(0:D) = \{0\}$ and hence D is a faithful R-module.

Conversely, assume M contains a nonzero faithful injective submodule D. If $A \neq 0$ is a left ideal of R then $Am \neq 0$ for some $m \in D$, since D is faithful, and so $\theta: A \to D$ defined by $\theta(x) = xm$ for all $x \in A$ yields a nonzero member of $\operatorname{Hom}_R (A, D)$. Now if $B \neq 0$ is a torsion-free R-module then, since $Z(R) = 0$, $\operatorname{Hom}_R (B, E(R)) \neq 0$, and so if $0 \neq \alpha \in \operatorname{Hom}_R (B, E(R))$ then $A = \alpha(B) \cap R \neq 0$. Then by injectivity of D, any nonzero map in $\operatorname{Hom} (A, D)$ extends to a nonzero map in $\operatorname{Hom}_R (\alpha(B), D)$ and this implies $\operatorname{Hom}_R(B, M) \neq 0$, completing the proof.

An interesting consequence of our previous results is the following

THEOREM 4. *Let R be a semiprime finite-dimensional torsion-free ring. Then $E(R)$ is a projective R-module if and only if R is a semisimple Artinian ring.*

PROOF. Let Q be a torsion-free injective R-module and F a free R-module mapping onto Q, say $\alpha: F \to Q$. Now $F \cong \oplus \sum_{\lambda \in \Lambda} R_\lambda$ where each $R_\lambda = R$ and there is a natural embedding of F in $\oplus \sum_{\lambda \in \Lambda} E(R_\lambda) = M$. Thus α extends to $\beta \in \operatorname{Hom}_R (M, Q)$. Since $E(R) = E(R_\lambda)$ is projective, M is a projective R-module. By Teply's result [9, Theorem 2.1], R has property (*) and so M is also an injective R-module. Thus by Lemma 1, Q is a direct summand of M and so Q is a projective R-module. Thus every torsion-free injective R-module is a torsionless R-module in the sense of Bass [2], and since submodules of torsionless R-modules are torsionless [2], every torsion-free R-module is a torsionless R-module. But this is equivalent to $\operatorname{Hom}_R (B, R) \neq 0$ for all torsion-free R-modules $B \neq 0$. By Theorem 3, R contains a faithful injective submodule and so the maximal injective submodule K of R is nonzero and faithful. From Theorem 2, $R = K \oplus S$ with K semisimple Artinian. Since $(SK)^2 = 0$ and $SK \subseteq K$ we have $SK = 0$ and so $S = 0$ since K is faithful. This completes the proof since the converse is trivial.

The condition that R be semiprime cannot be omitted. For if R is a left Artinian hereditary QF-3 ring (i.e. $E(R)$ is projective) then R is finite-dimensional and $Z(R)=0$ by [4, p. 426]. However, R need not be semisimple Artinian.

While prime rings are not necessarily finite-dimensional, a result similar to Theorem 3 holds for torsion-free prime rings. We first have

LEMMA 3. *A prime ring R is torsion-free if and only if R has a nonzero torsion-free R-module.*

PROOF. If $Z(R)=0$ then R is a torsion-free R-module. For the converse, we note that if $K\neq 0$ is an ideal of R then for any left ideal $A\neq 0$, $0\neq KA\subseteq K\cap A$, so K is an essential left ideal of R. Thus if $Z(R)\neq 0$ then $Z(R/Z(R))=R/Z(R)$ and this would imply every nonzero R-module has nonzero singular submodule.

THEOREM 5. *Let R be a prime ring having a nonzero torsion-free R-module M. Then* $\mathrm{Hom}_R(B, M)\neq 0$ *for all torsion-free R-modules $B\neq 0$ if and only if M contains a nonzero injective submodule.*

PROOF. By Lemma 3, $Z(R)=0$ hence $E(R)$ is torsion-free. Thus if M contains a nonzero image of any nonzero torsion-free R-module, M has a nonzero injective submodule as in Theorem 3. On the other hand let M contain a nonzero injective submodule D. For any torsion-free R-module $A\neq 0$, $(0:A)$ is an ideal of R, hence, as in Lemma 3, if $(0:A)\neq 0$ then $(0:A)$ is an essential left ideal of R; but then for any $0\neq a\in A$, $(0:A)\subseteq(0:a)$ and this contradicts $Z(A)=0$. Hence D is faithful and as in Theorem 3, $\mathrm{Hom}_R(B, M)\neq 0$ for all torsion-free $B\neq 0$.

Finally, we consider self-injective rings. In [6], Goldie defined for an R-module M, the sequence of submodules $Z_i(M)$, $i\geqq 0$, where $Z_0(M)=0$ and for $i\geqq 1$, $Z_i(M)$ is given by $Z_i(M)/Z_{i-1}(M)=Z(M/Z_{i-1}(M))$. Moreover, he established that $Z_i(M)=Z_2(M)$ for all $i\geqq 3$. Consequently for any R-module M, $Z_2(M)$ can have no essential extension in M. In particular if M is injective then $M=Z_2(M)\oplus N$; this has also been noted in [7].

THEOREM 6. *Let R be a self-injective ring. Then every finite-dimensional torsion-free R-module M is a completely reducible injective R-module.*

PROOF. Since M is finite-dimensional every nonzero submodule contains a nonzero uniform submodule, so let $S\neq 0$ be a uniform submodule of M. If $0\neq x\in S$ then $0\neq Rx\subseteq S$ is a torsion-free image of the injective R-module R and so, by Lemma 1, Rx is injective.

Since S is uniform and Rx is a direct summand of S, $Rx=S$. Thus any nonzero element of S generates S and so S is a simple injective R-module. Letting $M=S\oplus M_1$, if $M_1\neq 0$ we proceed as above and write $M_1=S_1\oplus M_2$ where S_1 is a simple injective R-module. Continuing in this manner and using the finite dimensionality of M, we arrive at the desired conclusion.

COROLLARY 6. *If R is a self-injective ring having property* (*), *then* $R=Z_2(R)\oplus K$ *(ring direct sum), where K is a semisimple Artinian ring. Hence every torsion-free R-module is a completely reducible injective R-module.*

PROOF. By the remarks preceding Theorem 6, $R=Z_2(R)\oplus K$. Since R satisfies (*) it has no infinite direct sum of torsion-free left ideals. Hence K is finite-dimensional and torsion-free and thus completely reducible. If $x\in Z_2(R)$ then Kx is a homomorphic image of K and so isomorphic to a direct summand of K. Since also $Kx\subseteq Z_2(R)$ we conclude that $Kx=0$. Now $Z_2(R)$ is an ideal of R and so $KZ_2(R)=Z_2(R)K=0$ and hence the decomposition is two-sided. That every torsion-free R-module is injective and hence completely reducible now follows from [**1**, Theorem 3.1].

Any quasi-Frobenius ring is both left Noetherian and self-injective, so that Corollary 6 provides an extension of [**7**, Proposition 2.8].

ADDED IN PROOF. We have noticed that R. N. Gupta, using different methods, has also obtained Theorem 4, see Osaka J. Math. **5** (1968), Theorem 4.1.

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UNIVERSITY OF TEXAS