ON THE EXTENSION OF LINEARLY INDEPENDENT SUBSETS OF FREE MODULES TO BASES

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Introduction. In this note we discuss a class of rings with identity with the following property:

(1) Each linearly independent subset of a (unitary) free right A-module can be extended to a basis, by adjoining elements of a given basis. In view of (1) we call such rings right-Steinitz rings. We prove the equivalence of (1) and the following condition:

(2) Let $R_1 = \{x \in A \mid x$ does not have a left inverse$\}$. If an infinite matrix T of elements of R_1 is column-finite and if $T_{ij} = 0$ for all $i \leqq j$, then, for each j, there is an integer N such that $(T + T^2 + \cdots + T^n)_{j+n,j} = 0$ for all $n > N$.

To prove the equivalence of (1) and (2) we need to establish several other properties of right-Steinitz rings, which in turn reveal them as being either examples or "near-examples" of classes of rings studied by a variety of investigators, the following cases being representative. In [1], P. M. Cohn discusses a sequence of three progressively stronger conditions, the strongest being

III. Any generating set with n elements of a rank n free module is free. An inductive argument shows that right-Steinitz rings do indeed satisfy the condition. It also follows from the discussion below that right-Steinitz rings satisfy all conditions of Goldie's local-rings except that the intersection of all powers of the ideal of nonunits may not be zero (cf., e.g., [2]). Obviously, division rings are right-Steinitz rings. If Z is the ring of integers and if p is any prime, then $Z/(p^i)$ satisfies condition (2) as is easily seen. For any field Δ and a vector-space V over Δ, let $A = \Delta \times V$, with operations defined by

$$(\delta_1, x_1) + (\delta_2, x_2) = (\delta_1 + \delta_2, x_1 + x_2)$$
$$(\delta_1, x_1)(\delta_2, x_2) = (\delta_1\delta_2, x_1\delta_2 + x_2\delta_1), \qquad \delta_i \in \Delta, \qquad x_i \in V.$$

Then, V is the ideal of nonunits, with $V^2 = 0$, and again condition (2) is easily seen to be satisfied. Another property of right-Steinitz rings is the following: if $\{x_i\}_{i=0}^{\infty}$ is a sequence of nonunits, then, for some index n, $x_n \cdot x_{n-1} \cdots x_1 = 0$. Thus, let F_0 be a division-ring, and let $F_0[x]$ be the polynomial-ring in one variable over F_0. Define $F_i = xF_0[x]/x^{i+1}F_0[x]$ for $i \geqq 1$. Let R be the weak direct sum of rings

Presented to the Society, January 23, 1969; received by the editors September 25, 1968.

F_i. Then, R is a two-sided vector-space over F_0. Take $A=F_0\times R$, with operations

$$(\delta_1, x_1)+(\delta_2, x_2) = (\delta_1+\delta_2, x_1+x_2),$$
$$(\delta_1, x_1)\cdot(\delta_2, x_2) = (\delta_1\delta_2, x_1\delta_2+\delta_1x_2+x_1x_2), \qquad \delta_i\in F_0, \qquad x_i\in R.$$

Notice that if $x_0=(\alpha, \cdots, \alpha_{i+1}, 0, \cdots, 0, \cdots)\in R$, then given any sequence $\{x_i\}_{i=0}^{\infty}$ of elements in R, $x_i\cdot x_{i-1}\cdot \cdots \cdot x_0=0$. From this, again, condition (2) follows. Notice that in this case the ideal R is not nilpotent, while if R is nilpotent, (2) follows easily.

Clearly, if T is an infinite proper triangular matrix, i.e., a triangular matrix with 0 diagonal, over any ring, then the inverse of $I-T$ exists and is equal to $I+T+T^2+\cdots$. The argument depends on the fact that $I-T$ as well as $I+T+T^2+\cdots$ are row finite and because $(T^n)_{ij}=0$ if $n>i-j$. We can thus restate condition (2) to obtain the equivalent form:

(2)′ If T is an infinite column-finite proper triangular matrix of elements of R_1, so is $(I-T)^{-1}$. In concluding this introduction we should like to thank the referee for several helpful comments and a simplification of the proof of Theorem 2.

The equivalence of conditions (1) and (2). Note that all modules under discussion are right unitary.

LEMMA 1. *If A satisfies* (1), *then for each infinite sequence $\{x_i\}_{i=0}^{\infty}$ of elements of A which do not have a left inverse, there is a nonnegative integer in such that $x_nx_{n-1}\cdots x_0=0$.*

PROOF. Let $\{u_i\}_{i=0}^{\infty}=U$, be a basis for a free A-module M, i.e $M=[U]=[U_i]_{i=0}^{\infty}$. Let $v_i=u_i-u_{i+1}x_i$, $i=0, 1, 2, \cdots$. Then, $\{v_i\}_{i=0}^{\infty}$ is linearly independent. Indeed, $\sum_{i=0}^{s} v_ia_i=0$ implies $\sum_{i=0}^{s}(u_i-u_{i+1}x_i)a_i=0$, i.e.,

$$u_0a_0+u_1(a_1-x_0a_0)+\cdots+u_s(a_s-x_{s-1}a_{s-1})-u_{s+1}x_sa_s=0,$$

whence $a_0=a_1=\cdots=a_s=0$. Now let V be the submodule spanned by $\{V_i\}_{i=0}^{\infty}$. Since $\{V_i\}_{i=0}^{\infty}$ can be extended to a basis of M by adjoining elements of U, suppose $\{v_i\}_{i=0}^{\infty}\cup\{u_{i_1}, u_{i_2}, \cdots\}$ is a basis of M. Then $u_{i_1}\equiv u_{i_2}y \pmod V$ for some $y\in A$ if $i_1<i_2$, whence $u_{i_1}\in\operatorname{span}(v, \{u_{i_2}\})$. Thus $\{v_i\}_{i=1}^{\infty}\cup\{u_t\}$ must be a basis for some $u_t\in U$. Then, $u_{t+1}\equiv u_ta \pmod V$, $u_t\equiv u_{t+1}x_t \pmod V$. Hence, $u_t\equiv u_tax_t \pmod V$, i.e., $1-ax_t=0$. Since x_t does not have a left inverse, $V=M$, and $\{v_i\}_{i=0}^{\infty}$ is a basis of M. Thus, if $\sum_{i=0}^{s} v_ib_i=u_0$, i.e.,

$$u_0b_0+u_1(b_1-x_0b_0)+\cdots+u_s(b_s-x_{s-1}b_{s-1})-u_{s+1}x_sb_s=u_0,$$

we have $b_0=1$, $b_1=x_0, \cdots$, $b_s=x_{s-1}\cdots x_0$, $x_sb_s=x_s\cdots x_0=0$. Hence, $n=n(s)=s$ and the lemma follows.

LEMMA 2. *Let A be a ring with identity, $R_1=\{x\in A \mid x$ does not have a left inverse$\}$ and $R_2=\{x\in A \mid x$ does not have a right inverse$\}$. If every element of R_1 is nilpotent then $R_1=R_2$ and R_1 forms the unique maximal ideal of A.*

PROOF. First we show that $R_1^c=A\backslash R_1$ forms a group. It is clear that R_1^c is closed under multiplication. Suppose $x\in R_1^c$, then there is a $y\in A$ such that $y\cdot x=1$. If $y\notin R_1^c$ then there is an integer n such that $y^n=0$, $y^{n-1}\neq 0$. Hence, $0=y^n\cdot x=y^{n-1}$, and this is a contradiction. So, $y\in R_1^c$. Therefore, if $yx=1$ then $xy=1$. Thus, $R_1^c\subset R_2^c$. Hence $R_1\supset R_2$. Suppose $x\in R_1$ and $x\notin R_2$, then there is a $y\in A$ such that $xy=1$. Since x is nilpotent, this is also a contradiction. Hence $R_1=R_2$. To show R_1 is closed under $+$, let x and y be elements of R_1, and suppose $x+y\in R_1$. Then there is a $z\in A$ such that $z(x+y)=1$, $zx+zy=1$, $zx=1-zy$. Since $zy\in R_1$, it is nilpotent and $1-zy$ has an inverse, i.e., zx has an inverse. This is a contradiction. Hence R_1 is closed under $+$. It is clear that $zR_1\subset R_1$ for any $z\in A$. Also if $x\in R_1$, $z\in A$ and $zx\notin R_1$, then there is a $y\in A$ such that $yxz=1$. This is a contradiction because yx is nilpotent. Hence R_1 is an ideal of A. It is clear that R_1 is the unique maximal ideal of A because R_1^c consists of the units of A. In short, since R_1 forms a left ideal, Lemma 2 follows as is well known.

COROLLARY 3. *If, for each infinite sequence $\{x_i\}_{i=0}^{\infty}$ of elements of A which do not have a left inverse there is a nonnegative integer n such that*

$$x_n\cdot x_{n-1}\cdots x_0=0,$$

then there is a nonzero element a of A such that $b\cdot a=0$ for all elements b of A which do not have a right inverse.

PROOF. Let R_2 be the collection of all nonzero elements of A which do not have a right inverse. If, for all nonzero elements x of R_2, $R_2x\neq\{0\}$, we have a choice function f: $R_2\backslash 0\to R_2\backslash 0$, such that $(x)f\cdot x\neq 0$, whence each nonzero x_1 in R_2 generates an infinite sequence $\{x_1,\cdots,x_n,\cdots\}$ with $x_i=(x_{i-1}\cdots x_1)f$, such that $x_n\cdot x_{n-1}\cdots x_1\neq 0$ for each integer n. This is a contradiction. Hence, since $R_2=\{0\}$ implies $R_2\cdot 1=0$, the lemma follows.

THEOREM 1. *If a ring A satisfies* (1), *then it satisfies* (2).

PROOF. From Lemma 1 and Lemma 2, A has a unique maximal ideal consisting of all nonunits, $R_1=R_2=R$. Let T be a matrix provided by (2), and let

$$u = \{u_j \mid j = 1, 2, \cdots\}$$

be a basis of a free A-module M. Let

$$v_j = u_j - \sum_l u_l T_{lj} \qquad \text{for } j = 1, 2, 3, \cdots,$$

then clearly $V = \{v_j \mid j = 1, 2, 3, \cdots\}$ is a linearly independent subset of M. From the corollary to Lemma 2, there is a nonzero element a such that $Ra = 0$. Hence $v_j a = U_j a$ for each j. Since we suppose that A satisfies (1) and V is a basis of M. Suppose $\sum_{j=1}^{i+n} v_j S_{ji} = u_i$ and $S_{ji} \in A$ for each i. Let S be the matrix whose elements are S_{ji}, then $(I-T)S = 1$ where $I_{ij} = \delta_{ij}$, where as mentioned before, $S = I + T + T^2 + \cdots$.

Lemma 3. *Let A be a ring satisfying condition* (2), *then for each sequence $\{x_i\}_{i=1}^{\infty}$ of elements of R_1, there is an n such that*

$$x_n \cdot x_{n-1} \cdots x_1 = 0.$$

Proof. Consider the case $T_{ij} = x_j$ if $i = j+1$ and $T_{ij} = 0$ if $i \neq j+1$. Then

$$(T^n)_{n,1} = T_{n+1,n} \cdot T_{n,n-1} \cdots T_{2,1} = x_n \cdot x_{n-1} \cdots x_1.$$

Hence from condition (2), $x_n \cdot x_{n-1} \cdots x_1 = 0$.

Lemma 4. *If A satisfies condition* (2) *then $R_1 = R_2 = R$ and if $R \neq \{0\}$ there is a nonzero element $a \in R$ such that $Ra = 0$, and R is the unique maximal ideal from Lemmas* 3, 2 *and the corollary to Lemma* 2.

Lemma 5. *Let A be a ring as in the corollary to Lemma* 2, *then any finite linearly independent subset of a free A-module M can be extended to a basis by adjoining elements of a given basis.*

Proof. Let $V = \{v_1, v_2, \cdots, v_n\}$ be a linearly independent set, and $U = \{u_i \mid i \in \Lambda\}$ be a basis of M. Let $v_1 = \sum u_i a_i$ for $a_i \in A$, then not all a_i are elements of R, otherwise $v_1 a = \sum u_i a_i a = 0$, where a is the element of A of Corollary 3. Let $a_1 \notin R$, then $u_1 = (v_1 - \sum_{i>1} u_i a_i) a_1^{-1}$, hence $\{v_1\} \cup \{u_i \mid i \neq 1\}$ is a basis. Suppose $\{v_1, v_2, \cdots, v_{n-1}\} \cup \{u_i \mid i > n\}$ is a basis and $v_n = \sum_{i<n} v_i b_i + \sum_{i \geq n} u_i a_i$, then not all a_i are in R, otherwise $v_n a = \sum_{i<n} v_i b_i a$. Hence $v_1, v_2, \cdots, v_n$ can be extended to a basis by adjoining some elements of U. Therefore, by induction, the lemma is proved.

Theorem 2. *If a ring A satisfies* (2) *then it satisfies* (1).

Proof. Let $U = \{u_i \mid i \in \Lambda\}$ be a basis of M, and $V = \{v_j \mid j \in T\}$ be a linearly independent subset. Without loss of generality we may assume that V is a maximal linearly independent subset of $V \cup U$. Suppose $[V] \neq M$, then there is a $u_1 \notin [V]$. Let $u_1 c = \sum_{j=1}^{n} V_j b_{j1}$ for some $c \in A$ and $b_{j1} \in A$. Since $\{v_1, v_2, \cdots, v_n\}$ can be extended to a basis by adjoining some elements of U,

$$u_1 = \sum_j v_j b'_{j1} + \sum_l u_l T_{l1} \quad \text{for some } b'_{j1}, \qquad T_{l1} \in A,$$

whence $b'_{j1} C = b_{j1}$ and $T_{l1} C = 0$ for all j and l. Hence $T_{l1} \in R$ and $u_1 \equiv \sum_{l \geqq 1} u_l T_{l1}$ mod $[V]$. If $T_{11} \neq 0$, then $u_1 \neq \sum_{l \geqq 2} u_l T_{l1}(1 - T_{11})^{-1}$ mod [V], and we may thus assume $u_1 \equiv \sum_{l \geqq 2} u_l T_{l1}$ mod $[V]$. Repeating this argument, we obtain a countably infinite column-finite matrix T of elements of R such that $T_{li} = 0$ if $l \leqq i$ and $u_i \equiv \sum_l u_l T_{li}$ mod $[V]$. By (2)′, $S = (I - T)^{-1}$ is column-finite. If X denotes the row matrix $(u_1, u_2, \cdots)$, then $X(I - T) \equiv 0$ mod $[V]$ implies $(X(I - T))S \equiv 0$ mod $[V]$, contradicting the fact that $u_1 \notin [V]$.

Corollary. *If a ring A satisfies* (2), *then, for any A-module M, $M = MR$ implies $M = \{0\}$.*

Proof. Let $\{u_i | i \in T\}$ be a generating set, then for each u_i, $u_i = \sum_l u_l T_{li}$ where $T_{li} \in R_1$. We can assume that $T = \{1, 2, \cdots\}$ and $T_{li} = 0$ if $l \geqq i$ as before. Then, $u_i - \sum_l u_l T_{li} = 0$ implies $u_i = 0$ for each i.

References

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