ON THE EXTENSION OF LINEARLY INDEPENDENT SUBSETS OF FREE MODULES TO BASES

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Introduction. In this note we discuss a class of rings with identity with the following property:

(1) Each linearly independent subset of a (unitary) free right $A$-module can be extended to a basis, by adjoining elements of a given basis. In view of (1) we call such rings right-Steinitz rings. We prove the equivalence of (1) and the following condition:

(2) Let $R_1 = \{x \in A \mid x$ does not have a left inverse\}. If an infinite matrix $T$ of elements of $R_1$ is column-finite and if $T_{ij} = 0$ for all $i \leq j$, then, for each $j$, there is an integer $N$ such that $(T + T^2 + \cdots + T^n)_{j+n,j} = 0$ for all $n > N$.

To prove the equivalence of (1) and (2) we need to establish several other properties of right-Steinitz rings, which in turn reveal them as being either examples or "near-examples" of classes of rings studied by a variety of investigators, the following cases being representative.

In [1], P. M. Cohn discusses a sequence of three progressively stronger conditions, the strongest being

III. Any generating set with $n$ elements of a rank $n$ free module is free. An inductive argument shows that right-Steinitz rings do indeed satisfy the condition. It also follows from the discussion below that right-Steinitz rings satisfy all conditions of Goldie's local-rings except that the intersection of all powers of the ideal of nonunits may not be zero (cf., e.g., [2]). Obviously, division rings are right-Steinitz rings. If $Z$ is the ring of integers and if $p$ is any prime, then $Z/(p^i)$ satisfies condition (2) as is easily seen. For any field $\Delta$ and a vector-space $V$ over $\Delta$, let $A = \Delta \times V$, with operations defined by

$$(\delta_1, x_1) + (\delta_2, x_2) = (\delta_1 + \delta_2, x_1 + x_2)$$

$$(\delta_1, x_1)(\delta_2, x_2) = (\delta_1\delta_2, x_1\delta_2 + x_2\delta_1), \quad \delta_i \in \Delta, \quad x_i \in V.$$ 

Then, $V$ is the ideal of nonunits, with $V^2 = 0$, and again condition (2) is easily seen to be satisfied. Another property of right-Steinitz rings is the following: if $\{x_i\}_{i=0}^n$ is a sequence of nonunits, then, for some index $n$, $x_n \cdot x_{n-1} \cdots x_1 = 0$. Thus, let $F_0$ be a division-ring, and let $F_0[x]$ be the polynomial-ring in one variable over $F_0$. Define $F_i = xF_0[x]/x^{i+1}F_0[x]$ for $i \geq 1$. Let $R$ be the weak direct sum of rings

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Then, $R$ is a two-sided vector-space over $F_0$. Take $A = F_0 \times R$, with operations

$$(\delta_1, x_1) + (\delta_2, x_2) = (\delta_1 + \delta_2, x_1 + x_2),$$
$$(\delta_1, x_1) \cdot (\delta_2, x_2) = (\delta_1\delta_2, x_1\delta_2 + \delta_1x_2 + x_1x_2), \quad \delta_i \in F_0, \quad x_i \in R.$$

Notice that if $x_0 = (\alpha, \cdots, \alpha_{i+1}, 0, \cdots, 0, \cdots) \in R$, then given any sequence $\{x_i\}_{i=0}^\infty$ of elements in $R$, $x_i \cdot x_{i-1} \cdots \cdots \cdot x_0 = 0$. From this, again, condition (2) follows. Notice that in this case the ideal $R$ is not nilpotent, while if $R$ is nilpotent, (2) follows easily.

Clearly, if $T$ is an infinite proper triangular matrix, i.e., a triangular matrix with 0 diagonal, over any ring, then the inverse of $I - T$ exists and is equal to $I + T + T^2 + \cdots$. The argument depends on the fact that $I - T$ as well as $I + T + T^2 + \cdots$ are row finite and because $(T^n)_{ij} = 0$ if $n > i - j$. We can thus restate condition (2) to obtain the equivalent form:

$$(2)' \text{ If } T \text{ is an infinite column-finite proper triangular matrix of elements of } R, \text{ so is } (I - T)^{-1}. \text{ In concluding this introduction we should like to thank the referee for several helpful comments and a simplification of the proof of Theorem 2.}$$

The equivalence of conditions (1) and (2). Note that all modules under discussion are right unitary.

**Lemma 1.** If $A$ satisfies (1), then for each infinite sequence $\{x_i\}_{i=0}^\infty$ of elements of $A$ which do not have a left inverse, there is a nonnegative integer in such that $x_n x_{n-1} \cdots x_0 = 0$.

**Proof.** Let $\{u_i\}_{i=0}^\infty = U$, be a basis for a free $A$-module $M$, i.e $M = [U] = [U_i]_{i=0}^\infty$. Let $v_i = u_i - u_{i+1} x_i$, $i = 0, 1, 2, \cdots$. Then, $\{v_i\}_{i=0}^\infty$ is linearly independent. Indeed, $\sum_{i=0}^\infty v_i a_i = 0$ implies $\sum_{i=0}^\infty (u_i - u_{i+1} x_i) a_i = 0$, i.e.,

$$u_0 a_0 + u_1 (a_1 - x_0 a_0) + \cdots + u_s (a_s - x_{s-1} a_{s-1}) - u_{s+1} x_s a_s = 0,$$

whence $a_0 = a_1 = \cdots = a_s = 0$. Now let $V$ be the submodule spanned by $\{V_i\}_{i=0}^\infty$. Since $\{V_i\}_{i=0}^\infty$ can be extended to a basis of $M$ by adjoining elements of $U$, suppose $\{v_i\}_{i=0}^\infty \cup \{u_i, u_{i+1}, \cdots\}$ is a basis of $M$. Then $u_{i+1} y (mod V)$ for some $y \in A$ if $i_1 < i_2$, whence $u_{i_1} \in \text{span}(v, \{u_{i_2}\})$. Thus $\{v_i\}_{i=1}^\infty \cup \{u_i\}$ must be a basis for some $u_i \in U$. Then, $u_{i+1} = u_i a (mod V)$, $u_i = u_{i+1} x_i (mod V)$. Hence, $u_i = u_i a x_i (mod V)$, i.e., $1 - ax_i = 0$. Since $x_i$ does not have a left inverse, $V = M$, and $\{v_i\}_{i=0}^\infty$ is a basis of $M$. Thus, if $\sum_{i=0}^\infty v_i b_i = u_0$, i.e.,

$$u_0 b_0 + u_1 (b_1 - x_0 b_0) + \cdots + u_s (b_s - x_{s-1} b_{s-1}) - u_{s+1} x_s b_s = u_0,$$
we have $b_0 = 1$, $b_1 = x_0$, $b_2 = x_{s-1} \cdots x_0$, $x_s b_s = x_s \cdots x_0 = 0$. Hence, $n = n(s) = s$ and the lemma follows.

**Lemma 2.** Let $A$ be a ring with identity, $R_1 = \{ x \in A \mid x$ does not have a left inverse $\}$ and $R_2 = \{ x \in A \mid x$ does not have a right inverse $\}$. If every element of $R_1$ is nilpotent then $R_1 = R_2$ and $R_1$ forms the unique maximal ideal of $A$.

**Proof.** First we show that $R_1^c = A \setminus R_1$ forms a group. It is clear that $R_1^c$ is closed under multiplication. Suppose $x \in R_1^c$, then there is a $y \in A$ such that $y \cdot x = 1$. If $y \in R_1^c$ then there is an integer $n$ such that $y^n = 0$, $y^{n-1} \neq 0$. Hence, $0 = y^n \cdot x = y^{n-1}$, and this is a contradiction. So, $y \in R_1^c$. Therefore, if $yx = 1$ then $xy = 1$. Thus, $R_1^c \subset R_2^c$. Hence $R_1 \supset R_2$. Suppose $x \in R_1$ and $x \in R_2$, then there is a $y \in A$ such that $xy = 1$. Since $x$ is nilpotent, this is also a contradiction. Hence $R_1 = R_2$. To show $R_1$ is closed under $+$, let $x$ and $y$ be elements of $R_1$, and suppose $x + y \in R_1$. Then there is a $z \in A$ such that $z(x + y) = 1$, $zx + zy = 1$, $zx = 1 - zy$. Since $zy \in R_1$, it is nilpotent and $1 - zy$ has an inverse, i.e., $zx$ has an inverse. This is a contradiction. Hence $R_1$ is closed under $+$. It is clear that $zR_1 \subset R_1$ for any $z \in A$. Also if $x \in R_1$, $z \in A$ and $zx \in R_1$, then there is a $y \in A$ such that $yxz = 1$. This is a contradiction because $yx$ is nilpotent. Hence $R_1$ is an ideal of $A$. It is clear that $R_1$ is the unique maximal ideal of $A$ because $R_1^c$ consists of the units of $A$. In short, since $R_1$ forms a left ideal, Lemma 2 follows as is well known.

**Corollary 3.** If, for each infinite sequence $\{ x_i \}_{i=0}^\infty$ of elements of $A$ which do not have a left inverse there is a nonnegative integer $n$ such that

$$x_n \cdot x_{n-1} \cdots x_0 = 0,$$

then there is a nonzero element $a$ of $A$ such that $b \cdot a = 0$ for all elements $b$ of $A$ which do not have a right inverse.

**Proof.** Let $R_2$ be the collection of all nonzero elements of $A$ which do not have a right inverse. If, for all nonzero elements $x$ of $R_2$, $R_3 x \neq \{ 0 \}$, we have a choice function $f: R_2 \setminus \{ 0 \} \rightarrow R_2 \setminus \{ 0 \}$, such that $(x)f \cdot x \neq 0$, whence each nonzero $x_1$ in $R_2$ generates an infinite sequence $\{ x_1, \cdots, x_n, \cdots \}$ with $x_1 = (x_{i-1} \cdots x_1)f$, such that $x_n \cdot x_{n-1} \cdots x_1 \neq 0$ for each integer $n$. This is a contradiction. Hence, since $R_2 = \{ 0 \}$ implies $R_2 \cdot 1 = 0$, the lemma follows.

**Theorem 1.** If a ring $A$ satisfies (1), then it satisfies (2).

**Proof.** From Lemma 1 and Lemma 2, $A$ has a unique maximal ideal consisting of all nonunits, $R_1 = R_2 = R$. Let $T$ be a matrix provided by (2), and let
be a basis of a free \( A \)-module \( M \). Let

\begin{equation}
 v_j = u_j - \sum_{i} u_i T_{ij} \quad \text{for} \quad j = 1, 2, 3, \cdots,
\end{equation}

then clearly \( V = \{ v_j | j = 1, 2, 3, \cdots \} \) is a linearly independent subset of \( M \). From the corollary to Lemma 2, there is a nonzero element \( a \) such that \( Ra = 0 \). Hence \( v_j a = U_j a \) for each \( j \). Since we suppose that \( A \) satisfies (1) and \( V \) is a basis of \( M \). Suppose \( \sum_{i} u_j S_{ji} = u_i \) and \( S_{ji} \subseteq A \) for each \( i \). Let \( S \) be the matrix whose elements are \( S_{ji} \), then \( (I - T) S = 1 \) where \( I_{ij} = \delta_{ij} \), where as mentioned before, \( S = I + T + T^2 + \cdots \).

**Lemma 3.** Let \( A \) be a ring satisfying condition (2), then for each sequence \( \{ x_i \}_{i=1}^{\infty} \) of elements of \( R_1 \), there is an \( n \) such that

\[ x_n \cdot x_{n-1} \cdots x_1 = 0. \]

**Proof.** Consider the case \( T_{ij} = x_j \) if \( i = j + 1 \) and \( T_{ij} = 0 \) if \( i \neq j + 1 \). Then

\[ (T^n)_{n,1} = T_{n+1,n} \cdot T_{n,n-1} \cdots T_{2,1} = x_n \cdot x_{n-1} \cdots x_1. \]

Hence from condition (2), \( x_n \cdot x_{n-1} \cdots x_1 = 0. \)

**Lemma 4.** If \( A \) satisfies condition (2) then \( R_1 = R_2 = R \) and if \( R \neq \{ 0 \} \) there is a nonzero element \( a \in R \) such that \( Ra = 0 \), and \( R \) is the unique maximal ideal from Lemmas 3, 2 and the corollary to Lemma 2.

**Lemma 5.** Let \( A \) be a ring as in the corollary to Lemma 2, then any finite linearly independent subset of a free \( A \)-module \( M \) can be extended to a basis by adjoining elements of a given basis.

**Proof.** Let \( V = \{ v_1, v_2, \cdots, v_n \} \) be a linearly independent set, and \( U = \{ u_i | i \in \Lambda \} \) be a basis of \( M \). Let \( v_1 = \sum u_i a_i \) for \( a_i \in A \), then not all \( a_i \) are elements of \( R \), otherwise \( v_1 a = \sum u_i a_i a = 0 \), where \( a \) is the element of \( A \) of Corollary 3. Let \( a_i \in R \), then \( u_1 = (v_1 - \sum_{i>1} u_i a_i) a_1^{-1} \), hence \( \{ v_1 \} \cup \{ u_i | i \neq 1 \} \) is a basis. Suppose \( \{ v_1, v_2, \cdots, v_{n-1} \} \cup \{ u_i | i > n \} \) is a basis and \( v_n = \sum_{i < n} v_i b_i + \sum_{i \geq n} u_i a_i \), then not all \( a_i \) are in \( R \), otherwise \( v_n a = \sum_{i < n} v_i b_i a \). Hence \( v_1, v_2, \cdots, v_n \) can be extended to a basis by adjoining some elements of \( U \). Therefore, by induction, the lemma is proved.

**Theorem 2.** If a ring \( A \) satisfies (2) then it satisfies (1).

**Proof.** Let \( U = \{ u_i | i \in \Lambda \} \) be a basis of \( M \), and \( V = \{ v_j | j \in T \} \) be a linearly independent subset. Without loss of generality we may
assume that $V$ is a maximal linearly independent subset of $V \cup U$. Suppose $[V] \neq M$, then there is a $u_1 \in [V]$. Let $u_1c = \sum_{j=1}^{n} V_jb_j$ for some $c \in A$ and $b_j \in A$. Since $\{v_1, v_2, \ldots, v_n\}$ can be extended to a basis by adjoining some elements of $U$,

$$u_1 = \sum_j v_jb_j' + \sum_l u_lT_{li} \text{ for some } b_j', \quad T_{li} \in A,$$

whence $b_j'C = b_j$ and $T_{li}C = 0$ for all $j$ and $l$. Hence $T_{li} \in R$ and $u_1 \equiv \sum_{t=1}^{i} u_1T_{li} \mod [V]$. If $T_{li} \neq 0$, then $u_1 \equiv \sum_{t=2}^{i} u_1T_{li}(1 - T_{li})^{-1} \mod [V]$, and we may thus assume $u_1 \equiv \sum_{t=2}^{i} u_1T_{li} \mod [V]$. Repeating this argument, we obtain a countably infinite column-finite matrix $T$ of elements of $R$ such that $T_{li} = 0$ if $l \leq i$ and $u_i \equiv \sum_{t=i}^{i} u_1T_{li} \mod [V]$. By (2)', $S = (I - T)^{-1}$ is column-finite. If $X$ denotes the row matrix $(u_1, u_2, \ldots)$, then $X(I - T) \equiv 0 \mod [V]$ implies $(X(I - T))S \equiv 0 \mod [V]$, contradicting the fact that $u_1 \in [V]$.

**Corollary.** If a ring $A$ satisfies (2), then, for any $A$-module $M$, $M = MR$ implies $M = \{0\}$.

**Proof.** Let $\{u_i | i \in I\}$ be a generating set, then for each $u_i$, $u_i = \sum_{t=i}^{i} u_1T_{ti}$ where $T_{ti} \in R_1$. We can assume that $T = \{1, 2, \ldots\}$ and $T_{li} = 0$ if $l \geq i$ as before. Then, $u_i - \sum_{t=i}^{i} u_1T_{ti} = 0$ implies $u_i = 0$ for each $i$.

**References**


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