THE RANK OF A FLAT MODULE[1]

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In this paper it is shown that flat modules are direct limits of free modules of finite rank. We say a flat module A has rank r if r is the least integer such that A can be represented as a direct limit of free modules of rank r. The flat modules of rank r are characterized.

1. R is a ring with unit and module means unital right R-module. A directed system of R-modules (C, θ, D) consists of a directed set D and a function which associates with each $\alpha \in D$ an R-module C_α and, with each pair $\alpha, \beta \in D$ for which $\alpha \leqq \beta$, a homomorphism $\theta_\alpha^\beta: C_\alpha \to C_\beta$ such that, for $\alpha < \beta < \gamma$ in D, $\theta_\beta^\gamma \theta_\alpha^\beta = \theta_\alpha^\gamma$ and, for each $\alpha \in D$, θ_α^α is the identity map on C_α. If (C, θ, D) is a directed system of R-modules let K be the submodule of ΣC_α generated by $\{x_\alpha - \theta_\alpha^\beta(x_\alpha)\}$. The exact sequence $0 \to K \to \Sigma C_\alpha \to A \to 0$ is called the exact sequence of the system. Clearly A is the direct limit of the system.

DEFINITION 1. A module K is said to be map-pure in C if K is a submodule of C and for each element k of K there is a map θ from C to K with $\theta(k) = k$.

LEMMA 1. *If K is map-pure in C and $k_1, k_2, \cdots, k_n$ is a finite set of elements of K then there is a map from C to K which leaves $k_1, k_2, \cdots, k_n$ fixed.*

PROOF. Since K is map-pure in C, the lemma is true for $n=1$. Proceeding by induction, let $k_1, k_2, \cdots, k_n$ be a set of n elements in K. Let θ_n be a map from C to K leaving k_n fixed. Then $k_1 - \theta_n(k_1)$, $k_2 - \theta_n(k_2), \cdots, k_n - \theta_n(k_n)$ is a set of $n-1$ elements of K, so by the induction assumption there is a map θ from C to K which leaves them fixed.

Now $1-(1-\theta)(1-\theta_n) = 1-1+\theta_n+\theta-\theta\theta_n = \theta_n+\theta-\theta\theta_n$ is a map from C to K and, since k_n is in the kernel of $1-\theta_n$ and, for $i=1, 2, \cdots, n-1$, $k_i - \theta_n(k_i)$ is in the kernel of $1-\theta$, it leaves $k_1, k_2, \cdots, k_n$ fixed.

PROPOSITION 1. *Let (C, θ, D) be a directed system of R-modules and let $0 \to K \to C \to A \to 0$ be its exact sequence. Then K is map-pure in C.*

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PROOF. It is sufficient to show that each of the generators of K can be left fixed by a homomorphism from ΣC_γ to K. Let $\alpha, \beta \in D$ with $\alpha < \beta$ and let $x_\alpha \in C$. Define ϕ_α: $C_\alpha \to K$ by $\phi_\alpha(y) = y - \theta_\alpha^\beta(y)$. For $\gamma \in D$, $\gamma \neq \alpha$, let ϕ_γ: $C_\gamma \to K$ be the zero map. This determines a map ϕ: $\Sigma C_\gamma \to K$ which leaves $x_\alpha - \theta_\alpha^\beta(x_\alpha)$ fixed.

If we restrict our attention to some family $\mathcal{C}$ of finitely generated modules and call a direct sum of modules from $\mathcal{C}$ a $\mathcal{C}$-free module, Proposition 1 says that, if A is a direct limit of $\mathcal{C}$-free modules, there is an exact sequence $0 \to K \to C \to A \to 0$ where C is $\mathcal{C}$-free and K is map-pure in C. In this context we have a converse.

PROPOSITION 2. *Let $\mathcal{C}$ be a family of finitely generated modules and let $0 \to K \to C \to A \to 0$ be an exact sequence where C is $\mathcal{C}$-free and K is map-pure in C. Then A is a direct limit of copies of C.*

PROOF. Let the finitely generated submodules of K be indexed by a set D. For each $\alpha \in D$, let j_α: $C \to C$ be such that j_α is the identity on K_α and $j_\alpha(C) = \overline{K}_\alpha$ is a finitely generated submodule of K. This is possible because K_α is in a finitely generated direct summand of C. Define a partial ordering on D by $\alpha \leqq \beta$ if and only if $\alpha = \beta$ or $\overline{K}_\alpha \subset K_\beta$. This makes D a directed set since if α and β are in D, $\overline{K}_\alpha + \overline{K}_\beta$ is finitely generated, say it is K_γ, and then $\alpha, \beta \leqq \gamma$.

For each $\alpha \in D$, let C_α be a copy of C. If $\alpha \leqq \beta$, define θ_α^β: $C_\alpha \to C_\beta$ by

$$\begin{aligned} \theta_\alpha^\beta &= 1 && \text{if } \alpha = \beta, \\ &= 1 - j_\beta && \text{if } \alpha < \beta. \end{aligned}$$

To see that this forms a directed system, we note that if $\alpha < \beta < \gamma$ and $x \in C_\alpha$ then $j_\gamma j_\beta(x) = j_\beta(x)$ since $j_\beta(x) \in \overline{K}_\beta \subset K_\gamma$ and so is left fixed by j_γ. Then $\theta_\beta^\gamma \theta_\alpha^\beta(x) = x - j_\beta(x) - j_\gamma(x) + j_\gamma j_\beta(x) = x - j_\gamma(x) = \theta_\alpha^\gamma(x)$.

For each $\alpha \in D$, let θ_α: $C_\alpha \to A$ be the projection of C onto A. These maps commute with the directed system since $\theta_\beta \theta_\alpha^\beta(x) = (x - j_\beta(x))$ mod $K = x$ (mod K) $= \theta_\alpha(x)$, for $\alpha < \beta$. To see whether A is the direct limit of this system we need only check two more things. First, that A is generated by the submodules $\theta_\alpha(C_\alpha)$ of A, which is trivial since each θ_α is onto. Secondly, that if $\theta_\alpha(x) = 0$, with $x \in C_\alpha$ for some α, then there is a $\beta > \alpha$ such that $\theta_\alpha^\beta(x) = 0$. But the kernel of θ_α is K so, if $\theta_\alpha(x) = 0$, x is in some finitely generated submodule K_β of K. If there is such a β with $\beta > \alpha$, then $\theta_\alpha^\beta(x) = x - j_\beta(x) = 0$. Otherwise α is the final element in D so $K = K_\alpha = \overline{K}_\alpha$, and j_α is projection of C onto its direct summand K. In this case let $\{C_i\}$ be a sequence of copies C. Then A is the direct limit of the system

$$C_1 \xrightarrow{1 - j_\alpha} C_2 \xrightarrow{1 - j_\alpha} C_3 \xrightarrow{1 - j_\alpha} \cdots .$$

The following is due to Villamayor [1].

PROPOSITION 3. *The right R-module A is flat if and only if whenever* $0 \to K \to F \to A \to 0$ *is exact with F free then K is map-pure in F.*

COROLLARY 1. *Every flat module is a direct limit of free modules.*

PROOF. This follows from Proposition 2.

Govorov [3] and Lazard [4] have also obtained this result. The following is a generalization of Theorem 2 in [4].

LEMMA 2. *Every module is a direct limit of finitely presented modules. Moreover, if A is a module and* $\mathcal{P}$ *is a family of finitely presented modules then every map from a finitely presented module to A factors through a module in* $\mathcal{P}$ *if and only if A is a direct limit of copies of modules in* $\mathcal{P}$.

PROOF. Let A be a right module and N a countable set. Let F be free on $A \times N$. Map F to A by mapping each generator to its first component. Consider the set consisting of all pairs (F_I, K) where I is a finite subset of $A \times N$, F_I is free on I and K is a finitely generated submodule of F_I which maps to zero in A. Define a partial order by $(F_I, K) \leqq (F_J, L)$ if and only if $I \subset J$ and $K \subset L$. This is clearly directed and A is the direct limit of the finitely presented modules F_I/K, where the maps are all canonical.

Suppose every map from a finitely presented module to A factors through a module in the family $\mathcal{P}$ of finitely presented modules. Then for each (F_I, K) we have a map $F_I/K \to P$, where $P \in \mathcal{P}$, and a map $P \to A$ such that $(F_I/K \to P \to A) = (F_I/K \to A)$. Let $0 \to H \to G \to P \to 0$ be a finite presentation of P. Let $x_1, \cdots, x_n$ be a basis for G and denote by p_i the image of x_i in P and by a_i the image of p_i in A. Let J be a subset of $A \times N$, disjoint from I, and consisting of, for each $i = 1, \cdots, n$, an element with first component a_i.

The map from F_J onto P thus determined, together with the map $(F_I \to P) = (F_I \to F_I/K \to P)$, determines a map from $F_I \oplus F_J$ onto P and the kernel L of this map is finitely generated since P is finitely presented. Also $(F_I \oplus F_J \to P \to A) = (F_I \oplus F_J \to A)$, so L maps to zero in A. Now $P = (F_I \oplus F_J)/L$ and $(F_I, K) \leqq (F_{I \cup J}, L)$ so the system has a cofinal subset whose elements are isomorphic to elements of $\mathcal{P}$ and clearly A is the direct limit of this cofinal system.

Conversely, suppose A is the direct limit of the directed system (P, θ, D). Let $0 \to H \to \Sigma P_\alpha \to A \to 0$ be the exact sequence of the system. Then, by Proposition 1, H is map-pure in P. For any (F_I, K) let $I = \{x_1, \cdots, x_n\}$ and let K be generated by $\sum_{i=1}^n x_i r_{ij}$, $j = 1, \cdots, m$. Denote the image of x_i under $F_I/K \to A$ by a_i and let p_i map to a_i

under $\Sigma P_\alpha \to A$. Then $\sum_{i=1}^n p_i r_{ij} = k_j$ is in H so there is a map $\theta: \Sigma P_\alpha \to H$ which leaves $k_1, k_2, \cdots, k_m$ fixed. Map F_I to ΣP_α by sending x_i to $p_i - \theta(p_i)$. We have

$$\sum_{i=1}^n (p_i - \theta(p_i)) r_{ij} = (1 - \theta)(k_j) = 0,$$

so $F_I/K \to A$ factors through ΣP_α:

$$(F_I/K \to A) = (F_I/K \to \Sigma P_\alpha \to A).$$

The image of F_I/K in ΣP_α is contained in a finite direct sum $P_{\alpha_1} + \cdots + P_{\alpha_r}$. Pick $\gamma > \alpha_1, \cdots, \alpha_r$.
Then

$$(P_{\alpha_1} + \cdots + P_{\alpha_r} \to A) = (P_{\alpha_1} + \cdots + P_{\alpha_r} \to P_\gamma \to A).$$

Therefore

$$\begin{aligned}(F_I/K \to A) &= (F_I/K \to \Sigma P_\alpha \to A)\\ &= (F_I/K \to P_{\alpha_1} + \cdots + P_{\alpha_r} \to A)\\ &= (F_I/K \to P_{\alpha_1} + \cdots + P_{\alpha_r} \to P_\gamma \to A)\end{aligned}$$

and $F_I/K \to A$ factors through P_γ.

COROLLARY 2. *Every flat module is a direct limit of free modules of finite rank.*

PROOF. If the flat module A is represented as a direct limit of a system of free modules, we can show as in the above proof that every map from a finitely presented module V to A can be factored through one of the free modules in the system. Since the image of V in this free module is finitely generated, the map also factors through a free submodule of finite rank. Now A is a direct limit of free modules of finite rank by Lemma 2.

DEFINITION 2. A flat module A has rank r if and only if it can be represented as a direct limit of free modules of rank less than or equal to r and r is the least integer which has this property.

THEOREM 2. *A flat module A has rank less than or equal to r if and only if every finitely generated submodule of A is contained in a submodule of A which can be generated by r elements.*

PROOF. Suppose A is a flat module whose rank is less than or equal to r. Say A is the direct limit of the system (F, θ, D) where each F_α, $\alpha \in D$, is free of rank less than or equal to r. Let B be a submodule of

A generated by $b_1, \cdots, b_n$. For each i, pick α_i such that $b_i \in \theta_{\alpha_i}(F_{\alpha_i})$. Let α be larger than each α_i, $i=1, \cdots, n$. Then B is contained in $\theta_\alpha(F_\alpha)$, which can be generated by r elements.

Conversely, let A be a flat module such that every finitely generated submodule is contained in a submodule of A which can be generated by r elements. We show that every map from a finitely presented module to A factors through a free module of rank r and then the theorem follows from Lemma 2.

Let $V \to A$ be a map from the finitely presented module V into A. By Theorem 1 of [4] there exists a factorization $V \to F \to A$ of $V \to A$ through a finite free module F. Let B be the image of F in A. The module B is contained in a submodule B' of A generated by r elements $b_1, \cdots, b_r$. Let $F' \to B'$ the map of the free module F' on $x_1, \cdots, x_r$ onto B', which maps x_i onto b_i, $i=1, \cdots, r$. Since $F' \to B'$ is onto and F free, $F \to B \to B'$ factors in $F \to F' \to B'$. Finally

$$(V \to A) = (V \to F \to B \to B' \to A) = (V \to F \to F' \to B' \to A)$$

with F' free of rank r. This completes the proof.

Clearly the rank of a finitely generated flat module A is $\mu(A)$, the least number of elements required to generate A. If A is a finitely generated module over an integral domain R with quotient field Q, $\dim_Q(A \otimes_R Q) \leqq \mu(A)$, with equality only when A is free. Hence our definition of rank does not necessarily agree with the usual one when R is an integral domain. It is easy to see that the two concepts do agree for flat modules of finite rank over principal ideal domains.

References

1. Stephen U. Chase, *Direct products of modules*, Trans. Amer. Math Soc. **97** (1960), 457–473. MR **22** #11017.

2. P. M. Cohn, *On the free product of associative rings*, Math. Z. **71** (1959), 380–398. MR **21** #5648.

3. V. E. Govorov, *On flat modules*, Sibirsk. Mat. Ž. **6** (1965), 300–304. (Russian) MR **30** #4799.

4. Daniel Lazard, *Sur les modules plats*, C.R. Acad. Sci. Paris **258** (1964), 6313–6316. MR **29** #5883.

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