A NOTE ON QUOTIENT SEMIRINGS

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In a previous article [1] Allen obtained an exact analogue of the fundamental homomorphism theorem for a certain class of semiring homomorphisms called maximal homomorphisms. The crux of the argument involved the notion of a Q-ideal, by which the semiring is partitioned into cosets modulo such an ideal, and the construction of a particular quotient semiring modulo a Q-ideal. This note gives necessary and sufficient conditions, in terms of homomorphisms, for an ideal to be a Q-ideal; shows that Q-ideals are included among the k-ideals considered in [4]–[7]; and deduces that the particular quotient semiring modulo a Q-ideal used in [1] actually coincides with the more familiar quotient structure first employed in [2]–[3]. We use without comment the definitions, terminology, and notation of [1].

The familiar construction of a quotient semiring modulo an ideal is as follows: given an ideal I of semiring $(S, +, \cdot)$ define a relation ρ by

$$\rho = \{(x, y) \in S \times S : x + i_1 = y + i_2 \quad \text{for some } i_1, i_2 \in I\}.$$

Then ρ is a congruence on both $(S, +)$ and $(S, \cdot)$ and under the usual operations of addition $(\oplus)$ and multiplication $(\circ)$ of congruence classes the ρ-classes become a semiring $(S/I, \oplus, \circ)$. The ρ-class containing $a \in S$ need not be the coset $a+I$. From [5] we note that I is contained in a ρ-class C_I which is the smallest k-ideal of S containing I and the zero of $(S/I, \oplus, \circ)$, that $(S/I, \oplus, \circ) = (S/C_I, \oplus, \circ)$, and that I is a ρ-class if and only if I is a k-ideal. (An ideal is a ***k-ideal*** if whenever $x+i \in I$, where $x \in S$ and $i \in I$, we have $x \in I$.)

THEOREM 1. *Let I be a Q-ideal of semiring $(S, +, \cdot)$ and define a relation η by*

$$\eta = \{(x, y) \in S \times S : x, y \in q + I \quad \textit{for some } q \in Q\}.$$

Then $\eta = \rho$ and I is the ρ-class containing zero.

PROOF. If $(x, y) \in \eta$, say $x = q + i_1$, $y = q + i_2$ for some $q \in Q$ and i_1, $i_2 \in I$, then $x + i_2 = (q + i_1) + i_2 = (q + i_2) + i_1 = y + i_1$ so that $(x, y) \in \rho$. Conversely, suppose $(x, y) \in \rho$, say $x + i_1 = y + i_2$ with i_1, $i_2 \in I$. Let $x = q_1 + i_3$ and $y = q_2 + i_4$ where q_1, $q_2 \in Q$ and i_3, $i_4 \in I$. Then

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$$x + i_1 = (q_1 + i_3) + i_1 = q_1 + (i_3 + i_1) = y + i_2 = (q_2 + i_4) + i_2 = q_2 + (i_4 + i_2).$$

But $q_1+(i_3+i_1)\in q_1+I$ and $q_2+(i_4+i_2)\in q_2+I$, whence $q_1=q_2$ and $q_1+I=q_2+I$. Thus $(x, y)\in\eta$.

Now there is a unique $q\in Q$ with $0\in q+I$, say $0=q+i$, so $q+I=q+0+I=q+q+i+I\subseteq q+q+I$. By Lemma 7 of [1] $q+q+I$ is contained in a unique coset $q'+I$ $(q'\in Q)$, which, in view of $q+I\subseteq q+q+I$, is $q+I$. Thus $q+q+I=q+I$ implies $q+q=q+i_1$, for some $i_1\in I$. Then $I=0+I=q+i+I\subseteq q+I$ and

$$q + I = q + 0 + I = q + q + i + I = q + i_1 + i + I = 0 + i_1 + I \subseteq I$$

which shows that I is the η-class $q+I$ containing 0.

Theorem 1 and the preceding remarks immediately give

COROLLARY 2. *A Q-ideal I of semiring $(S, +, \cdot)$ is a k-ideal and the zero of the quotient semiring $(S/I, \oplus, \circ)$.*

REMARK. From Theorem 1 it follows that there is at most one partition of S by cosets of I; an independent proof of this without introducing relation ρ is straightforward.

THEOREM 3. *Let I be a Q-ideal of a semiring $(S, +, \cdot)$, let $(S/I, \oplus, \circ)$ be the quotient semiring defined above, and let $(\{q+I\}_{q\in Q}, \oplus_Q, \odot_Q)$ be the quotient semiring of* [1]. *Then these two quotient semirings are equal.*

PROOF. The set of η-classes is precisely the set $\{q+I\}_{q\in Q}$ and $\oplus_Q$ and $\odot_Q$ are ordinary addition and multiplication of η-classes. Theorem 1 shows $\eta=\rho$.

REMARK. Theorem 3 shows that the isomorphism of Theorem 9 in [1] can be replaced by equality.

THEOREM 4. *An ideal I of a semiring $(S, +, \cdot)$ is a Q-ideal for some subset Q of S if and only if there is a semiring homomorphism $\phi: S\to T$ such that the inverse image of each $t\in T$ is a coset of I.*

PROOF. If I is a Q-ideal for the subset Q of S, the natural homomorphism $\phi: S\to S/I$ is such that $\phi^{-1}(q+I)=q+I$. Conversely, suppose $\phi: S\to T$ is such a semiring homomorphism. Define η on S by $\eta=\{(x, y)\in S\times S: x\phi=y\phi\}$. Then η is a congruence on $(S, +, \cdot)$ and each η-class is a coset of I. The partition of S consisting of the distinct η-classes makes I into a Q-ideal.

In [1] a homomorphism $\phi: S\to S'$ is called *maximal* if for each

$a \in S'$ there is an element $c_a \in \eta^{-1}(\{a\})$ such that $x + \ker \eta \subseteq c_a + \ker \eta$ for each $x \in \eta^{-1}(\{a\})$. Thus for each $a \in S'$, $\eta^{-1}(\{a\}) = c_a + \ker \eta$. Conversely, if $\eta: S \to S'$ is any homomorphism such that for each $a \in S'$, $\eta^{-1}(\{a\}) = c_a + \ker \eta$ for some $c_a \in S$ then η is maximal. In short, a *homomorphism* $\eta: S \to S'$ *is maximal if and only if the inverse image of each* $a \in S'$ *is a coset of* $\ker \eta$. Together with Theorem 4 this immediately gives (see Lemma 14 of [1])

COROLLARY 5. *The kernel of a maximal homomorphism is a Q-ideal.*

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