LOCAL COMPLEX ANALYTIC CURVES IN AN ANALYTIC VARIETY

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1. **Introduction.** In a recent paper [3], H. Whitney posed the following problems:

Let X be a complex analytic variety, and let y be a nonisolated point of X.

(1) Can we choose a neighborhood N of y such that for each point x of N there exists a connected one-dimensional complex analytic subvariety $C(x)$ of N that contains x and y and is regular at each of its points except possibly y [3, p. 214]?

(2) Furthermore, can we choose the one-dimensional subvarieties $C(x)$ as above so that any two distinct members of the family $\{C(x)\}$ intersect only at y [3, p. 231]?

In this paper, we give a short elementary solution to problem (1), using some methods of T. Bloom [1]. The solution to problem (2) with dim $X=2$ is a special case of a result of Bloom [1]. We do not know the answer to question (2) with dim $X>2$.

We use the following terminology:

An *analytic set* in a (reduced) complex analytic space X is a closed complex analytic subvariety of X. An analytic set is said to be *regular* if it contains no singular points (i.e., if it is a complex manifold). An *analytic curve* in X is a pure 1-dimensional analytic set in X; an *analytic hypersurface* is an analytic set of constant codimension 1. Other standard terminology used in this paper can be found in [2].

THEOREM 1. *Let X be a complex analytic space, Y an analytic set in X, and y a point of Y. Then there exists a neighborhood N of y in X such that for all $x\in N-Y$, we can find a (globally) irreducible analytic curve $C(x)$ in N such that*

(i) $x\in C(x)$,

(ii) $C(x)\cap Y=\{y\}$,

(iii) $C(x)-\{y\}$ *is regular.*

The affirmative answer to question (1) follows from Theorem 1 with $Y=\{y\}$. Note that it does not follow that $C(x)$ is locally irreducible at y, and it remains an open question whether we can extend Theorem 1 by adding the condition that $C(x)$ be irreducible at y.

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We shall show that Theorem 1 is a consequence of the following result.

THEOREM 2. *Let U be an open set in* $\mathbf{C}^{n+1}$ $(n \geqq 1)$, *and let A be an analytic hypersurface in U such that* $0 \in A$. *Then there exists a neighborhood $U' \subset U$ of 0 such that for all $p \in U' - A$, we can find a connected regular analytic curve $C(p)$ in U' with $p \in C(p)$ and $C(p) \cap A = \{0\}$.*

2. **Proof that Theorem 2 implies Theorem 1.** We use induction on $\dim_y X$. If $\dim_y X$ is 0 or 1, the theorem is trivial. So suppose Theorem 1 has been proven for dimensions up to and including n, and let $\dim_y X = n+1$. We assume without loss of generality that $\dim_y Y \leqq n$, since otherwise we can ignore those irreducible $(n+1)$-dimensional components of X which are contained in Y. By restricting our consideration to a sufficiently small neighborhood of y (which we also call X), we can assume that X has the following "local parametrization": $X = X_1 \cup Z$, where Z is an analytic variety of dimension at most n containing y, and X_1 is an analytic variety of pure dimension $n+1$ containing y; we furthermore have a neighborhood U of 0 in $\mathbf{C}^{n+1}$, an analytic hypersurface A in U, and a proper holomorphic map π: $X_1 \to U$ such that $\pi^{-1}(0) = \{y\}$, $\pi(X_1 \cap Z) \subset A$, and π is a local biholomorphism outside of $\pi^{-1}(A)$.

Note that $0 = \pi(y) \in A$ since $y \in X_1 \cap Z$. Let

$$T = \pi^{-1}(A) \subset X_1, \qquad X_2 = T \cup Z.$$

Therefore $X_1 \cup X_2 = X$, $X_1 \cap X_2 = T$, and $\dim X_2 = n$. Choose a neighborhood $U' \subset U$ of 0 as in the statement of Theorem 2, and let

$$N_1 = \pi^{-1}(U') \subset X_1.$$

Choose an open W in X such that $N_1 = X_1 \cap W$. By the induction hypothesis, we can choose a neighborhood $N_2 \subset X_2 \cap W$ of y that satisfies the conditions of Theorem 1 applied to $X_2 \cap W$. Let

$$N = (N_1 - T) \cup N_2 = W - (X_2 - N_2),$$

which is open in X and contains y.

In order to show that N satisfies the conditions of Theorem 1, it suffices to consider $x \in N_1 - T \subset N_1 - Y$. Then $\pi(x) \in U' - A$. Let L be a connected regular analytic curve in U' with $\pi(x) \in L$ and $L \cap A = \{0\}$, and let $C' = \pi^{-1}(L) \subset N_1$. Since $C' \cap T = \pi^{-1}(0) = \{y\}$, it follows that C' is an analytic curve in N, $C' - \{y\}$ is regular, and $C' \cap Y = \{y\}$. Let $C(x)$ be the irreducible component of C' that contains x. Since L is irreducible, $\pi(C(x)) = L$ and therefore $y \in C(x)$. Thus $C(x)$ is our desired analytic curve.

3. **Proof of Theorem** 2. We adopt the following convention throughout this section: If ξ is either a point in $\mathbf{C}^{n+1}$ or a function with values in $\mathbf{C}^{n+1}$, we write $\xi=(\xi^0, \xi^1, \cdots, \xi^n)$.

Let A and U be given as in the statement of Theorem 2. Shrink U if necessary, so that we can choose a holomorphic function f on U such that

$$A = \operatorname{loc}(f) = \{z \in U: f(z) = 0\}.$$

Make a linear change of coordinates (if necessary) so that 0 is an isolated point of $\operatorname{loc}(f, z^1, \cdots, z^n)$, where $z^0, z^1, \cdots, z^n$ are the coordinates in $U \subset \mathbf{C}^{n+1}$. Let

$$\pi = (f, z^1, \cdots, z^n): U \to \mathbf{C}^{n+1}.$$

Choose a neighborhood $V \subset \mathbf{C}^{n+1}$ of 0 and shrink U again so that $\pi(U) \subset V$, $\pi: U \to V$ is a proper map, and $\pi^{-1}(0)=\{0\}$ (see [2, p. 161]). Assume that $\partial f/\partial z^0$ vanishes at 0, since otherwise the conclusion of Theorem 2 would be obvious. Then $\operatorname{loc}(\partial f/\partial z^0)$ is an analytic hypersurface in U containing 0. Let $B=\pi(\operatorname{loc}(\partial f/\partial z^0))$, an analytic hypersurface in V. Choose a connected neighborhood $V' \subset V$ of 0 and a holomorphic function h on V' such that $B \cap V' = \operatorname{loc}(h)$. Choose open balls Δ and Δ' about 0 such that $\Delta' \subset\subset \Delta \subset\subset V'$. Let $U'=\pi^{-1}(\Delta')$, and note that

$$A \cap U' = \pi^{-1}\{w \in \Delta': w^0 = 0\}.$$

Let p be an arbitrary point in $U'-A$. Let $a=\pi(p) \in \Delta'$ (note that $a^0 \neq 0$), and let $L_a \subset \mathbf{C}^{n+1}$ be the complex line containing 0 and a. (The family of analytic curves $\{\pi^{-1}(L_a \cap \Delta')\}$ is a special case of a construction of Bloom [1].) In order to modify L_a so that we obtain (via π^{-1}) a regular analytic curve in U' that satisfies the conditions of the theorem, we need the following definition and lemma (which is proved at the end of this section).

Definition. If f is a holomorphic function defined in a neighborhood of a point $x \in \mathbf{C}^m$, we let $\nu(f; x)$ denote the order of f at x ($\nu(f; x) = 0$ if $f(x) \neq 0$; $\nu(f; x) = +\infty$ if $f \equiv 0$). Let $h \not\equiv 0$ be a holomorphic function on a connected open set $V \subset \mathbf{C}^{n+1}$. Let K be a closed subset of $\mathbf{C}$ without isolated points, and let $g: K \to V$ be holomorphic (i.e., g can be extended holomorphically to a neighborhood of K). We say that g is *h-transverse* if

$$\nu(h \circ g; t) = \nu(h; g(t)) \qquad \text{for all } t \in K.$$

(The condition that g be h-transverse means geometrically that for each point $t_0 \in \operatorname{loc}(h \circ g)$, the image under g_* of the tangent space of

$\mathbf{C}$ at t_0 is not contained in the tangent cone [3, pp. 211, 219–223] of loc (h) at $g(t_0)$.)

LEMMA. *Let $h \not\equiv 0$ be a holomorphic function on a connected open set $V \subset \mathbf{C}^{n+1}$, and let K be a connected compact subset of $\mathbf{C}$. Let c_1 and c_2 be distinct points of K, and let $w_1, w_2 \in V$. Consider the metric space $\mathfrak{F}$ (with the* sup-norm *metric) of all holomorphic maps $g = (g^0, \cdots, g^n): K \to V$ such that $g^0(t) = t$ and $g(c_j) = w_j$ for $j = 1, 2$. Suppose that $\mathfrak{F}$ is not empty. Then the set of h-transverse maps in $\mathfrak{F}$ is dense in $\mathfrak{F}$.*

Let $\psi: \mathbf{C} \to \mathbf{C}^{n+1}$ be the linear map given by $\psi(a^0) = a$ (and thus Image $(\psi) = L_a$). Let $K = \psi^{-1}(\bar{\Delta})$, a closed disk about 0; let

$$J = \{ra^0 : 0 \leqq r \leqq 1\} \subset K.$$

By applying the above lemma (with $c_1 = 0$, $c_2 = a^0$, $w_1 = 0$, $w_2 = a$), we can choose an h-transverse holomorphic map $g: K \to V'$ (near $\psi | K$) such that

(1) $g^0(t) = t$,
(2) $g(0) = 0$, $g(a^0) = a$,
(3) $g(J) \subset \Delta'$,
(4) $g(\partial K) \subset V' - \bar{\Delta}'$.

Let $t_1, \cdots, t_m \in K$ be the distinct zeros of $h \circ g$, and let $x_j = g(t_j) \in V'$, for $1 \leqq j \leqq m$. Let $\gamma(t)$ be a polynomial which vanishes to first order at $t_1, \cdots, t_m$, and a^0. (Note that $h \circ g(0) = 0$, so one of the t_j must be 0.) For $\lambda = (0, \lambda^1, \cdots, \lambda^n) \in \mathbf{C}^{n+1}$, define

$$g_\lambda(t) = g(t) + \gamma(t)\lambda.$$

For λ sufficiently small,

$$\nu(h \circ g_\lambda; t_j) \geqq \nu(h; x_j) = \nu(h \circ g; t_j),$$

and therefore $\nu(h \circ g_\lambda; t_j) = \nu(h \circ g; t_j)$ for $1 \leqq j \leqq m$, and $t_1, \cdots, t_m$ are the only zeros of g_λ in K. Thus, for such λ,

$$g_\lambda(K) \cap B = \{x_1, \cdots, x_m\}.$$

Let $C_\lambda = \pi^{-1}(g_\lambda(K) \cap \Delta')$. For λ small, C_λ is an analytic curve in U' that is regular outside of the finite set $S = \pi^{-1}\{x_1, \cdots, x_m\}$, since π has rank $n+1$ wherever $\partial f / \partial z^0 \neq 0$. Consider an arbitrary point $q \in S \cap U'$. Let

$$\left.\frac{\partial f}{\partial z^k}\right|_q = \beta_k, \qquad \left.\frac{\partial g_\lambda^k}{\partial t}\right|_{\pi(q)} = \alpha_\lambda^k, \qquad \text{for } 1 \leqq k \leqq n.$$

Since

$$C_\lambda = \mathrm{loc}(\overset{1}{z} - \overset{1}{g_\lambda} \circ f, \cdots, \overset{n}{z} - \overset{n}{g_\lambda} \circ f) \cap U',$$

it follows that C_λ is regular at q provided that the determinant

$$d = \det(\delta_j^k - \beta_j \overset{k}{\alpha_\lambda}) \qquad (1 \leqq j, k \leqq n)$$

does not vanish (where $\delta_j^k = 1$ if $k = j$, $\delta_j^k = 0$ if $k \neq j$). A simple calculation (for example, consider the characteristic polynomial of the matrix $(\beta_j \alpha_\lambda^k)$) shows that

$$d = 1 - \sum_1^n \beta_k \overset{k}{\alpha_\lambda}.$$

Since S is finite, we conclude that we can choose an arbitrarily small $\lambda = (0, \lambda^1, \cdots, \lambda^n)$ such that C_λ is regular. (One can also arrive at this conclusion, without calculating determinants, by instead proving a general fact about holomorphic maps from $U \subset \mathbf{C}^{n+1}$ into $\mathbf{C}^{n+1}$ that have rank n at a given point $q \in U$.) Choose a small λ such that g_λ satisfies conditions (1) through (4) above and C_λ is a regular analytic curve. Let $C(p)$ be the connected component of C_λ that contains p. Therefore, the analytic curve $\pi(C(p))$ equals the connected component of $g_\lambda(K) \cap \Delta'$ that contains a. Condition (3) above then implies that $0 \in \pi(C(p))$, and therefore $0 \in C(p)$. Thus $C(p)$ is our desired analytic curve.

To complete this discussion, we now prove the lemma: Let $\mathfrak{F}$, h, etc., be given as in the statement of the lemma. For $f \in \mathfrak{F}$, define

$$I(f; t) = \nu(h \circ f; t) - \nu(h; f(t)) \geqq 0,$$
$$I(f) = \sum I(f; t) \qquad (t \in K).$$

(The above sum is finite if $h \circ f \not\equiv 0$; $I(f) = +\infty$ if $h \circ f \equiv 0$.) Let $\mathfrak{F}_0$ be an arbitrary nonempty open subset of $\mathfrak{F}$. Choose a function $g \in \mathfrak{F}_0$ such that

$$I(g) = \min\{I(f) : f \in \mathfrak{F}_0\}.$$

We must show that $I(g) = 0$. Suppose, on the contrary, that $I(g) \geqq 1$. If $I(g) < +\infty$, let $t_1, \cdots, t_m \in K$ be the distinct zeros of $h \circ g$. Then

$$I(g) = \sum_1^m I(g; t_j).$$

Assume without loss of generality that $I(g; t_1) > 0$. Let $\gamma(t)$ be a polynomial that vanishes to first order at the points $c_1, c_2, t_1, \cdots, t_m$. For $\lambda = (0, \lambda^1, \cdots, \lambda^n) \in \mathbf{C}^{n+1}$, define

$$g_\lambda(t) = g(t) + \gamma(t)\lambda, \qquad \text{for } t \in K.$$

Thus $g_\lambda(t_j) = g(t_j)$ and $g_\lambda \in \mathfrak{F}_0$ for λ sufficiently small. By considering the Taylor expansion of $h \circ g_\lambda$ about t_1, we conclude that there exist arbitrarily small $\lambda = (0, \lambda^1, \cdots, \lambda^n)$ such that

$$\nu(h \circ g_\lambda; t_1) = \nu(h; g(t_1)) < \nu(h \circ g; t_1).$$

For such a λ sufficiently small, we let $Z_j \subset K$ denote the set of zeros of g_λ near t_j (for $1 \leqq j \leqq m$), and we conclude that

$$\sum (I(g_\lambda; t): t \in Z_j) \leqq I(g; t_j),$$

with the strict inequality holding for $j = 1$. Hence $I(g_\lambda) < I(g)$, contradicting the minimality of $I(g)$. Finally, if $I(g) = +\infty$, by repeating the above argument with $m = 1$ and t_1 an arbitrary point of K, we obtain g_λ with $I(g_\lambda) < +\infty$, also contradicting the minimality of $I(g)$.

References

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