COMMUTING TRANSFORMATIONS AND MIXING[1]

M. A. AKCOGLU, R. V. CHACON AND T. SCHWARTZBAUER

1. Introduction. The main purpose of this paper is to prove the following theorem: If two invertible measure preserving transformations commute, and the first admits an approximation by partitions, then the second is the limit of sums, composed of at most two terms, of positive and negative powers of the first.

The main theorem of the paper is a generalization of a result given in [1] where it is indicated that the result there can be applied to yield a simplification of the construction given in [2] of a transformation having continuous spectrum and no roots.

As an application, we show that the main result yields a way of constructing a transformation which is strongly mixing, and which therefore has continuous spectrum, and which commutes only with its powers (having, therefore, no roots). The striking result that such transformations exist was first obtained by Ornstein in [3].

2. Definitions and preliminaries. Let $(X, \mathfrak{F}, \mu)$ be a normalized nonatomic Lebesgue space (i.e., a space isomorphic to the unit interval). As usual, all sets that are referred to are understood to be in $\mathfrak{F}$, even if this is not explicitly stated.

DEFINITION 2.1. A collection ξ of sets having union $X_\xi \subset X$ will be called a partition if the sets are in $\mathfrak{F}$ and are pairwise disjoint. $\mathfrak{F}(\xi)$ will denote the σ-field of subsets of X_ξ generated by ξ. If $\xi = \{A_0, A_1, A_2, \cdots\}$ is a (finite or) countable partition, then any set $A \in \mathfrak{F}(\xi)$ can be written as $A = \sum_{i=0}^{\infty} a_i A_i$ where a_i, $i = 0, 1, \cdots$, is equal to zero or one, with the convention that $1B = B$, $0B = \emptyset$ for any set B.

DEFINITION 2.2. If ξ is a countable partition and $A \in \mathfrak{F}$, then among the sets in $\mathfrak{F}(\xi)$ there is at least one whose symmetric difference with A has minimal measure. We denote by $A(\xi)$ any one of these sets.

DEFINITION 2.3. If F, $F_n \in \mathfrak{F}$, we write $\lim_{n\to\infty} F_n = F$ provided that $\lim_{n\to\infty} \mu(F \Delta F_n) = 0$. If ξ_n is a sequence of countable partitions, we write $\xi_n \to \epsilon$ if for any $A \in \mathfrak{F}$, $\lim_{n\to\infty} A(\xi_n) = A$.

DEFINITION 2.4. Given an invertible and measure preserving transformation τ of X, we say that a partition ξ is τ-admissible if $\xi = \{A_0, A_1, \cdots, A_q\}$ is finite and $\tau A_{i-1} = A_i$, $1 \leq i \leq q$. The trans-

Received by the editors April 16, 1969.

[1] Research supported in part by NSF grant GP-3752 and by NRC grant A3974.

formation τ is said to have an approximation by partitions if there is a sequence $\xi_n = \{A_i(n): i=0, 1, \cdots, q(n)\}$ of τ-admissible partitions such that $\xi_n \to \epsilon$. If, in addition, $\lim_{n\to\infty} \mu(\tau A_{q(n)}(n) \Delta A_0(n))/\mu(A_0(n)) = 0$, then we say that τ has a strong approximation by partitions.

3. **Main result.** In what follows, τ and σ are two commuting invertible measure preserving transformations of X. For $A, B \in \mathfrak{F}$, we write $A = B + E(\epsilon)$ provided that $\mu(A \Delta B) \leqq \epsilon$.

LEMMA 3.1. *Let $\xi = \{A_0, A_1, \cdots, A_q\}$ be τ-admissible, $A, B \in \mathfrak{F}(\xi)$ and $\sigma A = B + E(\epsilon)$. Then there are two disjoint sets F and G and two nonnegative integers j, k, such that $\sigma A = \tau^j(A \cap F) + \tau^{-k}(A \cap G) + E(2\epsilon)$, $j + k \geqq q+1$ and $\mu(F) \leqq k/q+1$, $\mu(G) \leqq j/q+1$.*

PROOF. Let $C_i = A_0 \cap \sigma\tau^i A_0$, $-q \leqq i \leqq q$. If $x \in C_i \cap C_j$, then there exist two points $y, z \in A_0$ such that $x = \sigma\tau^i y = \sigma\tau^j z$, which implies that $\tau^{i-j} y = z$. Hence either $i = j$ or $|i-j| \geqq q+1$. Therefore any $q+1$ consecutive C_i's are pairwise disjoint.

Now, if we let $R = X - X_\xi$, then for any l, $0 \leqq l \leqq q$,

$$\sigma A_l = \sum_{k=0}^{q} A_k \cap \sigma A_l + R \cap \sigma A_l = \sum_{k=0}^{q} \tau^k C_{l-k} + R \cap \sigma A_l.$$

Hence, by writing

$$A = \sum_{l=0}^{q} a_l A_l, \qquad B = \sum_{k=0}^{q} b_k A_k = \sum_{k=0}^{q} \tau^k b_k A_0,$$

we obtain

$$\sigma A = \sum_{k=0}^{q} \tau^k \sum_{l=0}^{q} a_l C_{l-k} + R \cap \sigma A = \sum_{k=0}^{q} \tau^k b_k A_0 + E(\epsilon),$$

or

$$\sum_{k=0}^{q} \tau^k \sum_{l=0}^{q} a_l C_{l-k} = \sum_{k=0}^{q} \tau^k b_k A_0 + E(\epsilon).$$

Since for a fixed k, the sets C_{l-k}, $0 \leqq l \leqq q$, are pairwise disjoint, this means that

$$\sum_{k=0}^{q} \int_{A_0} \left| \sum_{l=0}^{q} a_l \Psi_{l-k} - b_k \right| d\mu < \epsilon,$$

where Ψ_i is the characteristic function of C_i.

Therefore, there exists a point $x_0 \in A_0$ such that

$$\sum_{k=0}^{q} \left| \sum_{l=0}^{q} a_l \Psi_{l-k}(x_0) - b_k \right| \mu(A_0) < \epsilon,$$

which means that $D = B + E(\epsilon)$ where

$$D = \sum_{k=0}^{q} \sum_{l=0}^{q} a_l \Psi_{l-k}(x_0) A_k.$$

Hence we have that $\sigma A = D + E(2\epsilon)$.

To see that this implies the lemma, we rearrange the double summation for D as follows:

$$D = \Psi_0(x_0)A + \sum_{j=1}^{q} \Psi_{-j}(x_0)\tau^j \sum_{l=0}^{q-j} a_l A_l + \sum_{k=1}^{q} \Psi_k(x_0)\tau^{-k} \sum_{l=k}^{q} a_l A_l,$$

where there are at most two nonzero terms. If there are less than two nonzero terms, we may define $F = \varnothing$ or $G = \varnothing$ or $F = G = \varnothing$ and see that all the conclusions of the lemma are satisfied. If there are exactly two nonzero terms, then

$$D = \tau^j \sum_{l=0}^{q-j} a_l A_l + \tau^{-k} \sum_{l=k}^{q} a_l A_l,$$

where $j, k \geqq 1$ and $j+k \geqq q+1$ since $C_{-j} \cap C_k \neq \varnothing$ implies $j+k \geqq q+1$ because any $q+1$ consecutive C_i's are pairwise disjoint. In this case we let $F = \sum_{l=0}^{q-j} A_l$, $G = \sum_{l=k}^{q} A_l$, and we have

$$\mu(F) \leqq \frac{q-j+1}{q+1} \mu(X_\xi) \leqq \frac{k}{q+1},$$

and similarly $\mu(G) \leqq j/q+1$.

We may now state and prove

Theorem 3.1. *If τ has an approximation by partitions and σ commutes with τ, then there are two sequences X_n^1, X_n^2 of sets and two sequences of nonnegative integers j_n^1, j_n^2, $n = 1, 2, \cdots$, such that $X_n^1 \cap X_n^2 = \varnothing$ and such that:*

(i) *if $A \in \mathfrak{F}$, then*

$$\sigma A = \lim_{n\to\infty} \left\{ \tau^{j_n^1}(A \cap X_n^1) + \tau^{-j_n^2}(A \cap X_n^2) \right\},$$

and

(ii) *either $\sigma = \tau^k$ for some integer k, or* $\lim_{n\to\infty} j_n^1 = \lim_{n\to\infty} j_n^2 = \infty$.

Proof. Let $\xi_n = \{A_0(n), A_1(n), \cdots, A_{q(n)}(n)\}$, $\xi_n \to \epsilon$, be a sequence of τ-admissible partitions. Note that $\lim_{n\to\infty} q(n) = \infty$. Choose a sequence $\epsilon_n>0$ such that $\epsilon_n q(n)\to 0$ as $n\to\infty$. For any integer $n=1, 2, \cdots$, we can find another integer $m(n)$ with the following properties:

(a) there exist B_n, $D_n\in\mathfrak{F}(\xi_{m(n)})$ such that $A_0(n)=B_n+E(\epsilon_n)$, $\sigma A_0(n)=D_n+E(\epsilon_n)$, and

(b) $q(n)\leqq\epsilon_n q(m(n))$.

We then have that $\sigma B_n=D_n+E(2\epsilon_n)$. Now choose j_n^1 and j_n^2 and F_n, G_n according to Lemma 3.1, with respect to the partition $\xi_{m(n)}$ and the set $B_n\in\mathfrak{F}(\xi_{m(n)})$. Hence, $\sigma B_n=\tau^{j_n^1}(B_n\cap F_n)+\tau^{-j_n^2}(B_n\cap G_n)+E(4\epsilon_n)$, and therefore

$$\sigma A_0(n) = \tau^{j_n^1}(A_0(n)\cap F_n) + \tau^{-j_n^2}(A_0(n)\cap G_n) + E(7\epsilon_n).$$

If we now let $X_n^1=\sum_{k=0}^{q(n)}\tau^k(A_0(n)\cap F_n)$ and $X_n^2=\sum_{k=0}^{q(n)}\tau^k(A_0(n)\cap G_n)$, we have that, for any $A\in\mathfrak{F}(\xi_n)$,

$$\sigma A = \tau^{j_n^1}(A\cap X_n^1) + \tau^{-j_n^2}(A\cap X_n^2) + E(7(q(n)+1)\epsilon_n),$$

and it is clear that this implies (i).

To prove (ii), note that, for example

$$\mu(X_n^2) \leqq (q(n)+1)\mu(G_n) \leqq (q(n)+1/q(n(m))+1)j_n^1,$$

which means that $\liminf_{n\to\infty}\mu(X_n^2)\leqq\liminf_{n\to\infty}(\epsilon_n j_n^1)$. Therefore if j_n^1 has a bounded subsequence, considering a subsequence if necessary, one may assume that $X_n^2=\varnothing$ and $j_n^1=k$ for all n.

As an application of this theorem, we give the following result.

THEOREM 3.2. *If a mixing transformation τ has an approximation by partitions, then it can only commute with its powers.*

Before the proof, we note that the nontrivial fact that there are mixing transformations that have an approximation by partitions follows from the results given in [3]. Hence Theorem 3.2 is not vacuously true.

PROOF. Using the notation of Theorem 3.1, we have $\sigma A\subset\tau^{j_n^1}A+\tau^{-j_n^2}A+E_n$ for all $n\geqq 1$ where $\mu(E_n)\to 0$. Hence

$$\sigma A\subset\left\{\tau^{j_n^1}A+\tau^{-j_n^2}A+E_n\right\}\cap\left\{\tau^{j_m^1}A+\tau^{-j_m^2}A+E_m\right\},$$

or

$$\sigma A\subset\tau^{j_n^1}A\cap\tau^{j_m^1}A+\tau^{j_n^1}A\cap\tau^{-j_m^2}A+\tau^{-j_n^2}A\cap\tau^{j_m^1}A$$
$$+\tau^{-j_n^2}A\cap\tau^{-j_m^2}A+E_{mn}$$

where $\mu(E_{mn})\to 0$ as m, $n\to\infty$. Now, if σ is not a power of τ, then j_n^1, $j_n^2\to\infty$ as $n\to\infty$. Hence, letting first $n\to\infty$ then $m\to\infty$, we see, from the mixing property,

$$\mu(\sigma A) = \mu(A) \leqq 4[\mu(A)]^2 \qquad \text{for all } A \in \mathfrak{F},$$

which is a contradiction.

LEMMA 3.2. *If, in addition to the hypotheses of Lemma* 3.1, $A_0=\tau A_q+E(\delta)$, *then there exists an integer* j, $-q\leqq j\leqq q$ *and a set* $A'\subset A$ *such that* $\sigma A=\tau^j A'+E(2\epsilon+2(q+1)\delta)$.

PROOF. We already have that $\sigma A=D+E(2\epsilon)$. If $D=\varnothing$ or D contains only one nonzero term, the lemma follows. Also, if $D=\tau^j\sum_{l=0}^{q-j}a_lA_l+\tau^{-k}\sum_{l=k}^{q}a_lA_l$ where $j+k=q+1$, we may write $\tau^{-k}A_l=\tau^j\tau^{-q-1}A_l=\tau^jA_l+E(\delta)$, which means that $D=\tau^jA+E((q+1)\delta)$, and again the lemma follows. Therefore, the only case which need be considered is the case for which $\Psi_{-j}(x_0)+\Psi_k(x_0)=2$ for some j, $k\geqq 1$ with $j+k>q+1$.

We will now give an upper bound for the measure of the set

$$S = \{x \mid \exists j,\, k,\ 1 \leqq j,\, k \leqq q,\ 1+q<j+k,\ \Psi_{-j}(x)+\Psi_k(x)=2\}.$$

By the definition of S,

$$\begin{aligned} S &= \sum_{j=2}^{q}\sum_{k=q-j+2}^{q} C_{-j}\cap C_k = \sum_{j=2}^{q}\sum_{k=q-j+2}^{q} A_0\cap\tau^{-j}\sigma A_0\cap\tau^k\sigma A_0 \\ &= \sum_{j=2}^{q}\tau^{-j}\left\{A_j\cap\sigma\left(A_0\cap\sum_{k=q-j+2}^{q}\tau^{k+j}A_0\right)\right\} \\ &= \sum_{j=2}^{q}\tau^{-j}\left\{A_j\cap\sigma\left(A_0\cap\sum_{l=2}^{j}\tau^{l}A_q\right)\right\} \end{aligned}$$

which means that $\mu(S)\leqq\mu(A_0\cap\sum_{l=1}^{q-1}\tau^{1+l}A_q)$. Hence $\mu(A_0\cap\tau A_q)>\mu(A_0)-\delta$ implies that $\mu(S)<\delta$.

Now going back to

$$\int_{A_0}\sum_{k=0}^{q}\left|\sum_{l=0}^{q}a_l\Psi_{l-k}-b_k\right|d\mu<\epsilon,$$

we see that there exists a point $x_0\in A_0-S$ such that

$$\sum_{k=0}^{q}\left|\sum_{l=0}^{q}a_l\Psi_{l-k}(x_0)-b_k\right|\mu(A_0)<\epsilon+(q+1)\delta,$$

since the integrand is bounded by $q+1$. This completes the proof.

The next theorem is proved in [1], but we include a simpler proof for the sake of completeness.

THEOREM 3.3. *If τ has a strong approximation by partitions and σ commutes with σ, then there is a sequence of nonnegative integers j_n, $n=1, 2, \cdots$, such that if $A \in \mathfrak{F}$ then*

$$\sigma(A) = \lim_{n\to\infty} \tau^{j_n}(A).$$

PROOF. The proof is the same as the proof of Theorem 3.1 with Lemma 3.2 replacing Lemma 3.1.

REFERENCES

1. R. V. Chacon and T. Schwartzbauer, *Commuting point transformations*, Z. Wahrscheinlichkeitstheorie und Verw. Gebeite **11** (1969), 277–287.

2. R. V. Chacon, *A geometric construction of measure preserving transformations*, Proc. Fifth Berkeley Sympos. Math. Statis. and Prob. (Berkeley, Calif., 1965/66) vol. II: Contributions to Prob. Theory, part 2, Univ. of California Press, Berkeley, Calif., 1967, pp. 335–360. MR **35** #3033.

3. D. Ornstein, *A mixing transformation that commutes only with its powers* (to appear).

UNIVERSITY OF MINNESOTA AND
UNIVERSITY OF TORONTO