SHORTER NOTES

The purpose of this department is to publish very short papers of an unusually elegant and polished character, for which there is no other outlet.

SIMPLICIAL AND PIECEWISE LINEAR COLLAPSIBILITY

MARSHALL M. COHEN[1]

We give a new proof of the following theorem. (The terminology is that of [2].)

If K and L are finite simplicial complexes such that $|K| \searrow |L|$ then there are subdivisions K_* and L_* such that $K_* \overset{s}{\searrow} L_*$.

Using this paper and [1] the foundations of p.l. topology can be presented without stellar subdivision.

LEMMA. *If $L < K$ and if there is a simplicial retraction $p: K \to L$ such that each nondegenerate point inverse is an arc then $K \overset{s}{\searrow} L$. (Note. "$<$" means "is a subcomplex of." All complexes are finite.)*

PROOF OF THE LEMMA. We proceed by induction on the number of simplexes A of L such that $f^{-1}(\hat{A}) \neq \hat{A}$. ($\hat{A}$ denotes the barycenter of A.) Let $A = A^n$ be of highest possible dimension among such simplexes. Then $f^{-1}(\hat{A})$ is an arc of the form $[a_0a_1] \cup \cdots \cup [a_{q-1}a_q]$, where $[a_{i-1}a_i] = f^{-1}(\hat{A}) \cap B_i$ for some $(n+1)$-simplex B_i of K and where a_{i-1}, a_i lie on the n-dimensional faces A_{i-1}, A_i of B_i. For some index, say j, $\hat{A} = a_j$. Then the arc traces out the *simplicial* collapse

$$K \searrow K - (\mathring{A}_0 \cup \mathring{B}_1) \searrow \cdots \searrow K - \bigcup_{i=0}^{j} (\mathring{A}_{i-1} \cup \mathring{B}_i)$$

$$\searrow K - \bigcup_{i=0}^{j} (\mathring{A}_{i-1} \cup \mathring{B}_i) - \bigcup_{i=0}^{q-j-1} (\mathring{A}_{q-i} \cup \mathring{B}_{q-i})$$

$$= (K - p^{-1}(\mathring{A})) \cup \mathring{A}.$$

The restriction of p to the latter complex satisfies the induction hypothesis, so this complex collapses simplicially to L.

PROOF OF THE THEOREM. Suppose that $|K| \searrow |L|$ by the elementary collapses

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$$|K| = |K_0| \searrow |K_1| \searrow \cdots \searrow |K_q| = |L|.$$

We may assume (by subdividing the K_i if necessary) that $K_{i+1} < K_i$ for all i. Let $K_i = K_{i+1} \cup B_i$ and $K_{i+1} \cap B_i = A_i$, where B_i is a ball and A_i is a face of B_i. By Newman's Theorem [1], $(B_i, A_i) \approx (I^{n_i} \times I, I^{n_i} \times 0)$. Hence there is a p.l. retraction $p_i: K_{i-1} \to K_i$ such that $p_i^{-1}(x)$ is a point or an arc for each $x \in |K_i|$. Choose [2, Theorem 1], subdivisions $S_i(K_i)$ so that all the maps $p_i: S_{i-1}(K_{i-1}) \to S_i(K_i)$ are simultaneously simplicial. Since $K_i < K_{i-1}$, $S_{i-1}(K_i)$ is a well-defined subcomplex of $S_{i-1}(K_{i-1})$. Since $p_i \| K_i | = 1$, $S_{i-1}(K_i) = S_i(K_i)$. Thus the lemma applies to each $p_i: S_{i-1}(K_{i-1}) \to S_i(K_i)$ and we have

$$K_* \equiv S_0(K_0) \overset{s}{\searrow} S_1(K_1) \overset{s}{\searrow} \cdots \overset{s}{\searrow} S_q(K_q) \equiv L_*.$$

References

1. M. Cohen, *A proof of Newman's theorem*, Proc. Cambridge Philos. Soc. **64** (1968), 961–963.

2. E. C. Zeeman, *Seminar on combinatorial topology*, Inst. Hautes Études Sci., Paris, 1963 (mimeographed).

Cornell University