SHORTER NOTES

The purpose of this department is to publish very short papers of an unusually elegant and polished character, for which there is no other outlet.

SIMPLICIAL AND PIECEWISE LINEAR COLLAPSIBILITY

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We give a new proof of the following theorem. (The terminology is that of [2].)

If $K$ and $L$ are finite simplicial complexes such that $|K| \nsubseteq |L|$ then there are subdivisions $K_*$ and $L_*$ such that $K_* \nsubseteq L_*.$

Using this paper and [1] the foundations of p.l. topology can be presented without stellar subdivision.

Lemma. If $L < K$ and if there is a simplicial retraction $p: K \to L$ such that each nondegenerate point inverse is an arc then $K \nsubseteq L.$ (Note. "\nsubseteq" means "is a subcomplex of." All complexes are finite.)

Proof of the lemma. We proceed by induction on the number of simplexes $A$ of $L$ such that $f^{-1}(A) \neq \hat{A}.$ (\hat{A} denotes the barycenter of $A$.) Let $A = A^n$ be of highest possible dimension among such simplexes. Then $f^{-1}(\hat{A})$ is an arc of the form $[a_0a_1] \cup \cdots \cup [a_{q-1}a_q],$ where $[a_{i-1}a_i] = f^{-1}(A) \cap B_i$ for some $(n+1)$-simplex $B_i$ of $K$ and where $a_{i-1}, a_i$ lie on the $n$-dimensional faces $A_{i-1}, A_i$ of $B_i.$ For some index, say $j,$ $\hat{A} = a_j.$ Then the arc traces out the simplicial collapse

$$K \nsubseteq K - (A_0 \cup B_1) \nsubseteq \cdots \nsubseteq K - \bigcup_{i=0}^{i} (A_{i-1} \cup B_i)$$

$$\nsubseteq K - \bigcup_{i=0}^{i} (A_{i-1} \cup B_i) - \bigcup_{i=0}^{q-1} (A_{q-i} \cup B_{q-i})$$

$$= (K - p^{-1}(\hat{A})) \cup \hat{A}.$$

The restriction of $p$ to the latter complex satisfies the induction hypothesis, so this complex collapses simplicially to $L.$

Proof of the theorem. Suppose that $|K| \nsubseteq |L|$ by the elementary collapses

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We may assume (by subdividing the $K_i$ if necessary) that $K_{i+1} \subset K_i$ for all $i$. Let $K_i = K_{i+1} \cup B_i$ and $K_{i+1} \cap B_i = A_i$, where $B_i$ is a ball and $A_i$ is a face of $B_i$. By Newman's Theorem [1], $(B_i, A_i) \simeq (I^n \times I, I^n \times 0)$. Hence there is a p.l. retraction $p_i : K_{i-1} \to K_i$ such that $p_i^{-1}(x)$ is a point or an arc for each $x \in K_i$. Choose [2, Theorem 1], subdivisions $S_i(K_i)$ so that all the maps $p_i : S_{i-1}(K_{i-1}) \to S_i(K_i)$ are simultaneously simplicial. Since $K_i \subset K_{i-1}$, $S_{i-1}(K_i)$ is a well-defined subcomplex of $S_{i-1}(K_{i-1})$. Since $p_i \| K_i \| = 1$, $S_{i-1}(K_i) = S_i(K_i)$. Thus the lemma applies to each $p_i : S_{i-1}(K_{i-1}) \to S_i(K_i)$ and we have

$$K_* = S_0(K_0) \bigtriangleup S_1(K_1) \bigtriangleup \cdots \bigtriangleup S_q(K_q) = L_*.$$

**References**


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