INJECTIVE HULLS OF STONE ALGEBRAS

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1. **Introduction.** R. Balbes and G. Grätzer [**1**] characterized injective Stone algebras. The purpose of this paper is to extend their results by characterizing injective hulls of Stone algebras (Theorem 3). In the process we also characterize essential extensions of Stone algebras (Theorem 1) and present a new proof of a characterization of injective Stone algebras given in [**1**]. The methods of this paper are based on the "triple" associated with a Stone algebra as developed by C. C. Chen and G. Grätzer [**3**].

2. **The triple.** In this section we recall several results of C. C. Chen and G. Grätzer [**3**]. Let $\langle S; \wedge, \vee, {}^*, 0, 1\rangle$, henceforth more briefly S, be a Stone algebra [2, p. 130]. The *center* of S, $C(S)$, is the set of complemented elements of S and so can be described as $C(S) = \{x^* \mid x \in S\}$; the set of *dense elements* $D(S)$ is $\{x \in S \mid x^* = 0\}$. Both $C(S)$ and $D(S)$ are sublattices of S, $C(S)$ is a Boolean algebra, and $D(S)$ is a distributive lattice with 1. A lattice homomorphism between lattices with 0, 1 that preserves 0 and 1 is said to be an *e-homomorphism*. Let $\mathfrak{D}(L)$ denote the lattice of dual ideals of the lattice L, ordered by set inclusion. With each Stone algebra S we associate an e-homomorphism $\phi(S): C(S) \to \mathfrak{D}(D(S))$ defined by

$$x\phi(S) = \{y \in D(S) \mid y \geqq x^*\}.$$

A *triple* $\langle C, D, \phi\rangle$ consists of a Boolean algebra C, a distributive lattice D with 1, and an e-homomorphism $\phi: C \to \mathfrak{D}(D)$. The main result of [**3**] is that there is a one-to-one correspondence between Stone algebras S and triples $\langle C, D, \phi\rangle$ such that $C = C(S)$, $D = D(S)$ and $\phi = \phi(S)$.

Let $\langle C, D, \phi\rangle$ and $\langle C_1, D_1, \phi_1\rangle$ be triples, let $f_1: C \to C_1$ be a homomorphism of Boolean algebras, and let $f_2: D \to D_1$ be a homomorphism of lattices preserving 1. The pair $\langle f_1, f_2\rangle$ is said to be a *homomorphism* from the triple $\langle C, D, \phi\rangle$ to the triple $\langle C_1, D_1, \phi_1\rangle$ if it satisfies the relation

$$x\phi f_2 \subseteq x f_1 \phi_1, \quad \text{for all } x \in C. \tag{1}$$

Let $\langle C, D, \phi\rangle$ be a triple, let I be an ideal of C, and let Θ be a congru-

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ence of D. The pair $\langle I, \Theta\rangle$ is said to be a *congruence* of the triple $\langle C, D, \phi\rangle$ if it satisfies the relation

$$x\phi \subseteq [1]\Theta, \quad \text{for all } x \in I. \tag{2}$$

($[1]\Theta$ denotes the congruence class of 1 under the congruence Θ.)

THEOREM A [3]. *The correspondence that assigns to f the pair $\langle f|C(S), f|D(S)\rangle$ is a one-to-one correspondence between homomorphisms $f: S\to S_1$ of Stone algebras and homomorphisms $\langle f_1, f_2\rangle$ of the corresponding triples. The homomorphism f is one-to-one (onto) if and only if both f_1 and f_2 are one-to-one (onto).*

COROLLARY. *The correspondence that assigns to a congruence Φ of a Stone algebra S the pair $\langle I, \Theta\rangle$, where I is the ideal determining $\Phi_{C(S)}$ (Φ restricted to $C(S)$) and Θ is $\Phi_{D(S)}$, is a one-to-one correspondence between congruences of Stone algebras and congruences of the corresponding triples.*

3. **Essential extensions.** Given any algebra A we use ω to denote the congruence on A determined by

$$x \equiv y(\omega) \quad \text{if and only if } x = y.$$

Let B be an algebra and let A be a subalgebra of B. The algebra B is said to be an *essential extension* of A if, for each congruence Θ of B, $\Theta_A = \omega$ implies $\Theta = \omega$.

If $\langle C, D, \phi\rangle$ is the triple associated with the Stone algebra S then the ideal $[1)\phi^{-1}$ of C is denoted by $R(S)$ and its pseudo-complementary ideal by $K(S)$; that is, $K(S) = \{x\in C \mid (x]\cap R(S) = (0]\}$.

THEOREM 1. *Let the Stone algebra S_1 be an extension of the Stone Algebra S. Then S_1 is an essential extension of S if and only if the following two conditions hold:*

(i) *the extension $D(S_1)$ of $D(S)$ is essential;*

(ii) *the extension $C(S_1)/K(S_1)$ of $C(S)/(K(S_1)\cap C(S))$ is essential.*

PROOF. We first establish the necessity of the two conditions. Let Θ be a congruence of $D(S_1)$ and let $\Theta_{D(S)} = \omega$. The pair $\langle (0], \Theta\rangle$ satisfies (2) and so determines a congruence Φ of S_1. By the second half of Theorem A, $\Phi_S = \omega$. Consequently $\Phi = \omega$; thus $\Theta = \omega$, establishing condition (i).

To establish condition (ii), let I be an ideal of $C(S_1)$ and let $I\cap C(S) \subseteq K(S_1)$. Let $J = I\cap R(S_1)$. The pair $\langle J, \omega\rangle$ satisfies (2) and so determines a congruence Φ on S_1. It follows that $\Phi_S = \omega$ since $J\cap C(S)$

$\subseteq K(S_1)\cap R(S_1)=(0]$. Thus $\Phi=\omega$, that is, $J=(0]$. Consequently $I\subseteq K(S_1)$, establishing condition (ii).

We now establish the sufficiency of the conditions. Let $\langle I, \Theta\rangle$ be a congruence of the triple $\langle C(S_1), D(S_1), \phi(S_1)\rangle$ and let $I\cap C(S)=(0]$, $\Theta_{D(S)}=\omega$. Then, by condition (i), $\Theta=\omega$. Thus, by (2), $I\subseteq R(S_1)$. Since $I\cap C(S)=(0]$ condition (ii) implies that $I\subseteq K(S_1)$; consequently $I\subseteq R(S_1)\cap K(S_1)=(0]$. The conclusion that $\langle I, \Theta\rangle=\langle(0], \omega\rangle$ establishes that S_1 is an essential extension of S, completing the proof of the theorem.

We now digress briefly to comment upon the relationship between $K(S)$ and $K(S_1)$ if S_1 is an arbitrary extension of S. In general there is no inclusion relation in either direction between $K(S)$ and $K(S_1)\cap C(S)$. We present two examples to illustrate this.

Let $S=\{0, 1\}$ and $S_1=\{0, a, 1\}$ be the two- and three-element chains. Both S and S_1 are Stone algebras and S_1 is an extension of S. Now $C(S)=C(S_1)=\{0, 1\}$, $K(S)=(0]$, and $K(S_1)=(1]$. Thus $K(S_1)\cap C(S)\not\subseteq K(S)$.

On the other hand, let S be represented by the triple $\langle C, D, \phi\rangle$ where C is the Boolean algebra $\mathbf{2}^2$, D is the two-element chain $\mathbf{2}$, and ϕ is defined as $\langle 0, x\rangle\phi=[1)$, $\langle 1, x\rangle\phi=[0)$, $x\in\{0, 1\}$. Let S_1 be represented by the triple $\langle C_1, D_1, \phi_1\rangle$ where $C_1=\mathbf{2}^3$, $D_1=\mathbf{2}$, and ϕ_1 is given by $\langle x, y, 0\rangle\phi_1=[1)$, $\langle x, y, 1\rangle\phi_1=[0)$, $x, y\in\{0, 1\}$. Let $i_1\colon C\rightarrow C_1$ be defined as $\langle 0, 0\rangle i_1=\langle 0, 0, 0\rangle$, $\langle 0, 1\rangle i_1=\langle 0, 1, 0\rangle$, $\langle 1, 0\rangle i_1=\langle 1, 0, 1\rangle$, $\langle 1, 1\rangle i_1=\langle 1, 1, 1\rangle$, and let $i_2\colon D\rightarrow D_1$ be the identity. The pair $\langle i_1, i_2\rangle$ is a one-to-one homomorphism of triples, $K(S_1)=(\langle 0, 0, 1\rangle]$, and $K(S)=(\langle 1, 0\rangle]$. Consequently $K(S_1)\cap C(S)=(\langle 0, 0\rangle]$, providing an example in which $K(S)\not\subseteq K(S_1)\cap C(S)$.

However, if S_1 is an essential extension of S we have

THEOREM 2. *Let the Stone algebra S_1 be an essential extension of the Stone algebra S. Then $K(S_1)\cap C(S)=K(S)$, and so $C(S_1)/K(S_1)$ is an essential extension of $C(S)/K(S)$.*

PROOF. Given a Stone algebra S and an element $a\in C(S)$ we recall the homomorphism $\rho_a\colon D(S)\rightarrow D(S)$ given by $x\rho_a=x\vee a^*$ (the join in S). The mapping ρ_a is a retraction onto $a\phi(S)$ and, since $x\vee a^*=x$ if and only if $x\vee a=1$ (a^* is the complement of a), ρ_a is the identity mapping if and only if $a^*\in R(S)$. If S_1 is an extension of S then ρ_a on $D(S_1)$ is an extension of ρ_a on $D(S)$ for all $a\in C(S)$. Consequently, if $D(S_1)$ is an essential extension of $D(S)$ then $R(S)\subseteq R(S_1)$. Thus $K(S_1)\cap C(S)\subseteq K(S)$.

Now let I be the ideal in $C(S_1)$ generated by $K(S)$. Then

$$I\cap R(S_1)\cap C(S)\subseteq R(S)\cap K(S)=(0]$$

since $R(S_1)\cap C(S)\subseteq R(S)$ whether or not the extension S_1 is essential. Thus, by condition (ii) of Theorem 1, $I\cap R(S_1)\subseteq K(S_1)$ and so $I\subseteq K(S_1)$. Consequently, if S_1 is an essential extension of S then $K(S)=K(S_1)\cap C(S)$. The rest of the theorem follows from Theorem 1.

4. The injective hull. An extension of an algebra is said to be an *injective hull* of the algebra if it is an essential injective extension. It follows from general considerations that any two injective hulls of a algebra are isomorphic over the algebra; we shall consequently refer to "the" injective hull of an algebra.

Let B be a Boolean algebra. The Stone algebra $B^{[2]}$ was introduced by R. Balbes and G. Grätzer [1]. $B^{[2]}$ is the sublattice of B^2 consisting of those pairs $\langle x, y\rangle$ such that $x\leqq y$; in $B^{[2]}$ $\langle x, y\rangle^*=\langle y', y'\rangle$. There are mappings p_1, p_2: $B^{[2]}\rightarrow B$, where $\langle x_1, x_2\rangle p_i=x_i$. The mapping p_1 is an e-homomorphism and p_2 is a homomorphism of Stone algebras.

We shall need the following result of [1]. We present a new proof that seems to be more direct than the original.

LEMMA 1 [1]. *Let B_0, B_1 be complete Boolean algebras. Then $B_0\times B_1^{[2]}$ is an injective Stone algebra.*

PROOF. Let S be a Stone algebra, let B be a Boolean algebra, and let f: $S\rightarrow B$ be an e-homomorphism of lattices. The mapping $f^\sigma:S\rightarrow B^{[2]}$, defined by $xf^\sigma=\langle xf, x^{**}f\rangle$, is then a homomorphism of Stone algebras. If $F:S\rightarrow B^{[2]}$ is a homomorphism of Stone algebras then $F=(Fp_1)^\sigma$, and if f: $S\rightarrow B$ is an e-homomorphism of lattices then $f=f^\sigma p_1$. The correspondence σ thus establishes a one-to-one correspondence between e-homomorphisms from S to B and homomorphisms of Stone algebras from S to $B^{[2]}$.

Now let B be a complete Boolean algebra, let $F:S\rightarrow B^{[2]}$ be a homomorphism of Stone algebras, and let S_1 be an extension of S. Since, by P. R. Halmos [5, p. 143] and the results of G. Grätzer and E. T. Schmidt [4], B is an injective distributive lattice, the e-homomorphism $F\circ p_1:S\rightarrow B$ has an extension $f_1:S_1\rightarrow B$. The homomorphism of Stone algebras $f_1^\sigma:S_1\rightarrow B^{[2]}$ is then an extension of F; thus $B^{[2]}$ is injective.

If B is a complete Boolean algebra then the homomorphism of Stone algebras $p_2:B^{[2]}\rightarrow B$, along with the embedding of B into $B^{[2]}$ that sends $x\in B$ to $\langle x, x\rangle$, establishes that B is isomorphic to a retract of $B^{[2]}$ and so is itself an injective Stone algebra.

The lemma now follows, since the direct product of injective algebras is injective.

We also need the following lemma.

LEMMA 2. *Let D be a distributive lattice with* 1, *let D^d be the lattice of* *principal dual ideals of D, and let $\mathfrak{D}'$ be the sublattice of $\mathfrak{D}(D)$ (the lattice of dual ideals) generated by D^d and the center of $\mathfrak{D}(D)$. Then $\mathfrak{D}'$ is an essential extension of D^d.*

PROOF. Let Θ be a congruence of $\mathfrak{D}'$ whose restriction to D^d is ω. Let $d_1, d_2 \in \mathfrak{D}'$, let $d_1 \leqq d_2$, and let $d_1 \equiv d_2(\Theta)$. Let $x \in d_2$. Then $[x) = [x) \wedge d_2 \equiv [x) \wedge d_1(\Theta)$. As observed in [3], the intersection of a complemented dual ideal and a principal dual ideal is principal. Thus, by the distributivity of $\mathfrak{D}(D)$, $\mathfrak{D}' = \{ [x) \vee d \mid x \in D,\ d$ in the center of $\mathfrak{D}(D) \}$. Consequently $[x) \wedge d_1$ is principal, and so $[x) = [x) \wedge d_1$. We conclude that $x \in d_1$ and thus $d_1 = d_2$, demonstrating that $\Theta = \omega$. The lemma is consequently proved.

THEOREM 3. *Let S be a Stone algebra. The injective hull of S is isomorphic to $B_0 \times B_1^{[2]}$ where B_0 is the completion of $C(S)/K(S)$ and B_1 is the completion of the Boolean algebra generated by $D(S)$.*

PROOF. For the sake of brevity we write the triple associated with S as $\langle C, D, \phi \rangle$. The triple associated with $B_0 \times B_1^{[2]}$ is $\langle B_0 \times B_1, B_1, \phi_1 \rangle$ where $\langle x_0, x_1 \rangle \phi_1 = [x_1')$. We construct an embedding $\langle i_1, i_2 \rangle$ of $\langle C, D, \phi \rangle$ into $\langle B_0 \times B_1, B_1, \phi_1 \rangle$. The e-homomorphism i_2 is the embedding of D into B_1.

We now define i_1. B_1 is the injective hull of D and so B_1^d, the dual of B_1, is the injective hull of D^d. Let $f: \mathfrak{D}' \to B_1^d$ be defined by $df = \bigvee (di_2)$, the $\bigvee$ taken in B_1^d. Since $f \mid D^d$ is one-to-one, Lemma 2 implies that f is one-to-one; consequently, $f': \mathfrak{D}' \to B_1$, defined by $df' = (df)' = (\bigwedge (di_2))'$ ($\bigwedge$ in B_1), is one-to-one. Let $q_1: C \to B_1$ be defined by $xq_1 = (\bigwedge (x\phi i_2))'$. Since $x\phi$ is in the center of $\mathfrak{D}(D)$ it follows that $xq_1 = x\phi f'$ ad thus $0q_1^{-1} = R(S)$. Let $q_0: C \to B_0$ be the composition of the natural homomorphism $C \to C/K(S)$ and the embedding of $C/K(S)$ into B_0; then $0q_0^{-1} = K(S)$. We define $i_1: C \to B_0 \times B_1$ by requiring that $xi_1 = \langle xq_0, xq_1 \rangle$. Since $0q_0^{-1} \cap 0q_1^{-1} = K(S) \cap R(S) = (0]$, it follows that i_1 is an embedding of Boolean algebras.

Let $x \in C$; then $xi_1\phi_1 = [(xq_1)') = [\bigwedge (x\phi i_2))$. Thus $x\phi i_2 \subseteq xi_1\phi_1$, and we conclude that $\langle i_1, i_2 \rangle$ is a homomorphism of triples. Since both i_1 and i_2 are one-to-one, $\langle i_1, i_2 \rangle$ yields an embedding of S into $B_0 \times B_1^{[2]}$.

We now observe that

$$K(B_0 \times B_1^{[2]}) = 0 \times B_1 \subseteq B_0 \times B_1.$$

For each $x \in C$, $xi_1 \in K(B_0 \times B_1^{[2]})$ if and only if $xq_0 = 0$, that is, if and only if $x \in K(S)$. Thus

$$K(B_0 \times B_1^{[2]}) \cap Ci_1 = K(S)i_1,$$

and, by the definition of B_0, it follows that $B_0 = (B_0 \times B_1)/K(B_0 \times B_1^{[2]})$ is an essential extension of $Ci_1/(K(B_0 \times B_1^{[2]}) \cap Ci_1)$. Since, as observed above, B_1 is an essential extension of D we conclude by Theorem 1 that $B_0 \times B_1^{[2]}$ is an essential extension of S under the embedding determined by $\langle i_1, i_2 \rangle$. The theorem then follows by Lemma 1.

COROLLARY [1]. *A Stone algebra is injective if and only if it is isomorphic to $B_0 \times B_1^{[2]}$, where B_0 and B_1 are complete Boolean algebras.*

BIBLIOGRAPHY

1. R. Balbes and G. Grätzer, *Injective and projective Stone algebras*, Duke Math. J. (to appear).

2. G. Birkhoff, *Lattice theory*, 3rd ed., Amer. Math. Soc. Colloq. Publ., vol. 25, Amer. Math. Soc., Providence, R. I., 1967. MR **37** #2638.

3. C. C. Chen and G. Grätzer, *Stone lattices*. I: *Construction theorems*, Canad. J. Math. 21 (1969), 884–894.

4. G. Grätzer and E. T. Schmidt, *On the generalized Boolean algebra generated by a distributive lattice*, Nederl. Akad. Wetensch. Proc. Ser. A.61 = Indag. Math. 20 (1958), 547–553. MR 20 #6991.

5. P. R. Halmos, *Lectures on Boolean algebras*, Van Nostrand, Math. Studies, no. 1, Van Nostrand, Princeton, N. J., 1963. MR 29 #4713.

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