APPLICATIONS OF AN INEQUALITY FOR THE SCHUR COMPLEMENT

EMILIE V. HAYNSWORTH

1. Introduction. Suppose B is a nonsingular principal submatrix of an $n\times n$ matrix A. The Schur Complement of B in A, denoted by (A/B), is defined as follows: Let $\hat{A}$ be the matrix obtained from A by the simultaneous permutation of rows and columns which puts B into the upper left corner of $\hat{A}$,

$$A = \begin{pmatrix} B & C \\ D & G \end{pmatrix},$$

leaving the rows and columns of B and G in the same increasing order as in A. Then the Schur Complement of B in A is

$$(A/B) = G - DB^{-1}C. \tag{1}$$

Schur proved that the determinant of A is the product of the determinant of any nonsingular principal submatrix B with that of its Schur complement,

$$|A| = |B| \, |(A/B)|. \tag{2}$$

The inertia of an Hermitian matrix A is given by the ordered triplet, In $A=(\pi, \nu, \delta)$, where π denotes the number of positive, ν the number of negative, and δ the number of zero roots of the matrix A. In a previous paper [2], it was shown that the inertia of an Hermitian matrix can be determined from that of any nonsingular principal submatrix together with that of its Schur complement. That is, if A is Hermitian, and B is a nonsingular principal submatrix of A, then

$$\text{In } A = \text{In } B + \text{In}(A/B). \tag{3}$$

More recently, the author, with Douglas Crabtree [1], proved the identity,

$$(A/B) = ((A/C)/(B/C)).$$

In Theorem 1 of §2 we make use of (3) to prove an extension of a theorem by Marcus [3]. Then in Theorem 2 we apply the result of Theorem 1 to obtain an inequality for the Schur complement which is similar to Minkowski's famous inequality (see [4]) for the determinant of the sum of positive definite Hermitian matrices:

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$$|A+B|^{1/n} \geqq |A|^{1/n} + |B|^{1/n} \quad \text{(Minkowski).}$$

This, of course, implies

$$(4) \qquad |A+B| \geqq |A| + |B|.$$

A number of extensions of the Minkowski inequality have been proved by Marcus, Minc and others (see [5]).

In Theorem 3 we obtain some new inequalities for the determinant of the sum of two positive definite Hermitian matrices.

2. **An extension of a theorem by Marcus.** In a recent paper [3] M. Marcus proved a number of interesting inequalities for positive definite Hermitian matrices, including the following: If H and K are positive definite matrices of order n, and X and Y are arbitrary vectors, then

$$(H^{-1}X, X) + (K^{-1}Y, Y) \geqq ((H+K)^{-1}(X+Y), (X+Y)).$$

It is shown in Theorem 1 that by making use of the properties of the Schur complement this inequality can be extended to the case where X and Y are arbitrary $n \times m$ matrices. We shall use the notation $A \geqq 0$ for a positive semidefinite matrix (p.s.d. matrix), with strict inequality implying that A is positive definite (p.d.). If A and B are p.s.d. matrices, the statement $A \geqq B$ will mean $A - B \geqq 0$.

THEOREM 1. *Suppose H and K are positive definite matrices of order n. Then if X and Y are arbitrary $n \times m$ matrices, the $m \times m$ matrix*

$$(5) \qquad Q = X^*H^{-1}X + Y^*K^{-1}Y - (X+Y)^*(H+K)^{-1}(X+Y)$$

is positive semidefinite.

PROOF. Let A and B be the Hermitian matrices of order $2n$,

$$A = \begin{pmatrix} H & X \\ X^* & X^*H^{-1}X \end{pmatrix}, \qquad B = \begin{pmatrix} K & Y \\ Y^* & Y^*K^{-1}Y \end{pmatrix}.$$

From (3), it is clear that a nonzero Hermitian matrix is positive semidefinite (definite) if and only if there exists a positive definite principal submatrix whose Schur complement is positive semidefinite (definite). Thus, by inspection, the matrices A and B are positive semidefinite. Then, since the sum of any two positive semidefinite matrices is also positive semidefinite (or definite) we have

$$A + B = \begin{pmatrix} H+K & X+Y \\ X^*+Y^* & X^*H^{-1}X + Y^*K^{-1}Y \end{pmatrix} \geqq 0.$$

This proves the theorem, as the matrix Q in (5) is the Schur complement of $H+K$ in $A+B$.

3. An inequality for the Schur complement.

THEOREM 2. *Suppose A and B are Hermitian matrices of order n, partitioned into 2×2 block matrices, $A=(A_{ij})$, $B=(B_{ij})$, $i, j=1, 2$, where A_{11} and B_{11} are square of order m. If $A\geqq 0$, $B\geqq 0$, $A_{11}>0$, $B_{11}>0$, then*

$$(A + B/A_{11} + B_{11}) \geqq (A/A_{11}) + (B/B_{11}). \tag{6}$$

PROOF. By the previous arguments, $A_{11}+B_{11}>0$, and $A+B\geqq 0$. From the definition,

$$(A + B/A_{11} + B_{11}) = (A_{22} + B_{22}) - (A_{21} + B_{21})(A_{11} + B_{11})^{-1} \cdot (A_{12} + B_{12}).$$

By Theorem 1,

$$(A_{21} + B_{21})(A_{11} + B_{11})^{-1}(A_{12} + B_{12}) \leqq A_{21}A_{11}^{-1}A_{12} + B_{21}B_{11}^{-1}B_{12}.$$

Thus

$$\begin{aligned}(A + B/A_{11} + B_{11}) &\geqq (A_{22} + B_{22}) - (A_{21}A_{11}^{-1}A_{12} + B_{21}B_{11}^{-1}B_{12}) \\ &= (A/A_{11}) + (B/B_{11}).\end{aligned}$$

This proves the formula (6), which we now apply to find a new inequality for the determinant of the sum of two positive definite Hermitian matrices.

4. Some determinantal inequalities.

THEOREM 3. *Suppose A and B are positive definite Hermitian matrices. Let A_k and B_k, $k=1, \cdots, n$, denote the principal submatrices of order k in the upper left corner of the matrices A and B respectively. Then*

$$|A + B| \geqq |A|\left(1 + \sum_{k=1}^{n-1} \frac{|B_k|}{|A_k|}\right) + |B|\left(1 + \sum_{k=1}^{n-1} \frac{|A_k|}{|B_k|}\right). \tag{7}$$

COROLLARY. *If A and B are positive definite, and $A>B$, then*

$$|A + B| > |A| + n|B|. \tag{8}$$

For the proof of Theorem 3 we need the following lemmas. Lemma 1 is probably well known, as it follows immediately from the Minkowski inequality (4). Lemma 2 follows as a corollary to Lemma 1 and Theorem 2.

LEMMA 1. *If A and B are positive definite Hermitian matrices and $A>B$, then $|A_k|>|B_k|$, $k=1, \cdots, n$.*

PROOF. Let $A-B=C>0$. Then $A_k=B_k+C_k$ $(k=1, \cdots, n)$ where A_k, B_k, and C_k are positive definite, since they are principal submatrices of positive definite matrices. Then by (4), $|A_k| \geqq |B_k|+|C_k| >|B_k|$ $(k=1, \cdots, n)$.

LEMMA 2. *If A and B satisfy the conditions of Theorem 2, then*

$$|(A+B/A_{11}+B_{11})| \geqq |A|/|A_{11}| + |B|/|B_{11}|.$$

PROOF. By Theorem 2 and Lemma 1,

$$\begin{aligned} |(A+B/A_{11}+B_{11})| &\geqq |(A/A_{11})+(B/B_{11})| \\ &\geqq |(A/A_{11})| + |(B/B_{11})| && \text{by (4)} \\ &= |A|/|A_{11}| + |B|/|B_{11}| && \text{by (2).} \end{aligned}$$

PROOF OF THEOREM 3. We prove the theorem by induction on n. For $n=2$, we have from (2),

$$|A+B| = |A_1+B_1| \, |(A+B/A_1+B_1)|. \tag{9}$$

By Lemma 2,

$$|(A+B/A_1+B_1)| \geqq |A|/|A_1| + |B|/|B_1|.$$

Thus, using (4) on the first factor on the right in (9),

$$|A+B| \geqq (|A_1| + |B_1|)(|A|/|A_1| + |B|/|B_1|)$$

which proves (7) for $n=2$.

Now assume (7) holds for matrices of order less than or equal to $n-1$. Then, if A and B are of order n,

$$|A+B| \geqq (|A_{n-1}+B_{n-1}|)|(A+B/A_{n-1}+B_{n-1})|,$$

where, by the inductive assumption,

$$\begin{aligned} &|A_{n-1}+B_{n-1}| \\ &\qquad \geqq |A_{n-1}|\left(1+\sum_{k=1}^{n-2}\frac{|B_k|}{|A_k|}\right) + |B_{n-1}|\left(1+\sum_{k=1}^{n-2}\frac{|A_k|}{|B_k|}\right), \end{aligned}$$

and, by Lemma 2,

$$|(A+B/A_{n-1}+B_{n-1})| \geqq |A|/|A_{n-1}| + |B|/|B_{n-1}|.$$

Thus

$$
\begin{aligned}
|A+B| &\geqq \left(|A_{n-1}|\left(1+\sum_{k=1}^{n-2}\frac{|B_k|}{|A_k|}\right)\right.\\
&\qquad \left.+|B_{n-1}|\left(1+\sum_{k=1}^{n-2}\frac{|A_k|}{|B_k|}\right)\right)\left(\frac{|A|}{|A_{n-1}|}+\frac{|B|}{|B_{n-1}|}\right)\\
&= |A|\left(1+\sum_{k=1}^{n-2}\frac{|B_k|}{|A_k|}\right)+|B|\left(1+\sum_{k=1}^{n-2}\frac{|A_k|}{|B_k|}\right)\\
&\qquad +\left|\frac{A_{n-1}}{B_{n-1}}\right|\left(1+\sum_{k=1}^{n-2}\frac{|B_k|}{|A_k|}\right)|B|\\
&\qquad +\left|\frac{B_{n-1}}{A_{n-1}}\right|\left(1+\sum_{k=1}^{n-2}\frac{|A_k|}{|B_k|}\right)|A|\\
&\geqq |A|\left(1+\sum_{k=1}^{n-1}\frac{|B_k|}{|A_k|}\right)+|B|\left(1+\sum_{k=1}^{n-1}\frac{|A_k|}{|B_k|}\right).
\end{aligned}
$$

This proves Theorem 3.

The corollary follows as an immediate consequence of Lemma 1, since if $A > B$,

$$|A_k|/|B_k| > 1 \qquad (k = 1, \cdots, n).$$

Hence

$$|A+B| \geqq |A|\left(1+\sum_{k=1}^{n-1}\frac{|B_k|}{|A_k|}\right)+n|B| \geqq |A| + n|B|.$$

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References

1. D. Crabtree and E. Haynsworth, *An identity for the Schur complement of a matrix*, Proc. Amer. Math. Soc. **22** (1969), 364–366.

2. E. Haynsworth, *Determination of the inertia of a partitioned Hermitian matrix*, Linear Algebra and Appl. **1** (1968), no. **1**, 73–81. MR **36** #6440.

3. M. Marcus, *An inequality for linear transformations*, Proc. Amer. Math. Soc. **18** (1967) 793–797. MR **36** #3811.

4. M. Marcus and H. Minc, *A survey of matrix theory and matrix inequalities*, Allyn & Bacon, Boston, Mass., 1964, p. 115. MR **29** #112.

5. M. Marcus and S. Pierce, *Extensions of the Minkowski inequality*, Linear Algebra and Appl. **1** (1968), no. **1**, 13–27. MR **36** #6431.

Auburn University