JORDAN'S THEOREM FOR SOLVABLE GROUPS

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ABSTRACT. We show that every finite solvable group of $n\times n$ matrices over the complex numbers has a normal abelian subgroup of index $\leqq 2^{4n/3-1}3^{10n/9-1/3}$. For infinitely many n, this bound is best possible.

Let C denote the complex numbers. A celebrated theorem of Jordan [9] states that there is a function $f(n)$, defined on the positive integers, such that any finite subgroup of $\mathrm{GL}(n, C)$ has an abelian normal subgroup of index $\leqq f(n)$. Two essentially different proofs of this theorem are familiar. One, found in §36 of [3] and due to Frobenius and Schur, provides an estimate for $f(n)$ on the order of $e^{n^2 \log n}$. The other [1], [2] shows that $f(n)$ is not essentially larger than $e^{n^2/\log n}$ (using the prime number theorem). The symmetric group on n letters has order $n!>e^{n \log n-n}$ and, being doubly transitive, has a faithful complex representation of degree $n-1$; thus we see $f(n)>e^{n \log n-n}$ in general. There is a large gap between these two bounds, and R. Brauer has asked (unpublished) if this gap can be narrowed.

In this paper we use methods of Dixon [5] to completely close the gap for the corresponding function $f_s(n)$ associated with solvable groups, and thus [7] we settle the case of groups of odd order. We are particularly indebted to Dixon for a personal communication which was of help in obtaining the best possible result. We prove the following

THEOREM. *Let n be a positive integer, F an algebraically closed field, G a completely reducible finite solvable subgroup of* $\mathrm{GL}(n, F)$. *Then there is an abelian normal subgroup A of G such that*

$$|G:A| \leqq 2^{4n/3-1}3^{10n/9-1/3} < e^{2.145n}.$$

Moreover, when $F=C$ and $n=3\cdot 4^k$, $k=0, 1, \cdots$, this bound is attained.

PROOF. Let $F(G)$ be the Fitting subgroup of G. We will use a theorem of Dixon [5], which asserts that

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$$|G:F(G)| \leqq 2^{n-1}3^{(2n-1)/3}.$$

To study $F(G)$, we first assume G is irreducible and primitive as a linear group. $F(G)$ is the direct product of its Sylow p-subgroups $O_p(G)$, for various primes p. A theorem of Blichfeldt, (Theorem (50.7) of [3]) shows that normal abelian subgroups of G are contained in the center $Z(G)$ of G, and Schur's lemma shows that $Z(G)$ is cyclic, consisting of scalar matrices only. Therefore characteristic abelian subgroups of $O_p(G)$ are cyclic, and a theorem of P. Hall (Theorem III.3.10 of [8]) describes the possible structure of $O_p(G)$.

Clifford's theorem, (Theorem (49.2) of [3]) asserts, since G is primitive, that the underlying vector space V of G is a direct sum of isomorphic irreducible $O_p(G)$-modules. These must each be faithful, so by Theorem V.5.17 of [8], $1 \neq O_p(G)$ implies $p \neq \text{char } F$.

If p is odd, P. Hall's theorem shows that $O_p(G) = E_p \circ Z_p$ (central product), where E_p is extra special of order p^{2n_p+1} (or $E_p = 1$), and Z_p is cyclic. All faithful irreducible representations of $O_p(G)$ have degree p^{n_p} (see [10]), so we have shown $p^{n_p} | n$; in particular, if $p > n$ then $O_p(G) = Z_p \subseteq Z(G)$. $Z_p = Z(O_p(G)) \subseteq Z(G)$ in any case, so

$$|O_p(G):O_p(G) \cap Z(G)| = p^{2n_p}.$$

If $p = 2$, then $O_p(G) = E_2 \circ B_2$ (central product), where E_2 is extra special or 1, and B_2 is cyclic, dihedral, semidihedral or generalized quaternion. We claim we may assume B_2 is cyclic. If not, and $|B_2| = 8$, then B_2 may be taken into E_2. If B_2 is noncyclic and $|B_2| \geqq 16$, then $\mho^1(O_2(G)) = \langle x^2 | x \in O_2(G) \rangle$ is a cyclic noncentral normal subgroup of G, a contradiction. So $B_2 = Z_2$ is cyclic, and the same considerations as for p odd show that if $|E_2| = 2^{2n_2+1}$, then $2^{n_2} | n$ and

$$|O_2(G):O_2(G) \cap Z(G)| = 2^{2n_2}.$$

Hence we conclude that $\prod_p p^{n_p}$ divides n and $|F(G):Z(G)| = \prod_p p^{2n_p}$ divides n^2. Thus

$$|G:Z(G)| \leqq 2^{n-1}3^{(2n-1)/3}n^2.$$

Our theorem is true for $n = 1$. If $n \geqq 2$ but $n \neq 3$, then $n^2 < 2^{n/3} \cdot 3^{4n/9}$, so the first part of the theorem holds when G is primitive and $n \neq 3$.

If G is primitive and $n = 3$, then $F(G) = P \circ Z(G)$ (central product), where P is extra special of order 27 and exponent 3, and P is characteristic in G. Hence $G/Z(G)$ is represented faithfully as an automorphism group of P fixing $Z(P) = P \cap Z(G)$ elementwise; $\text{Aut}(P)$ is well known, and we see $|G:Z(G)| \leqq 216$, proving the first part of the theorem in this case $n = 3$, G primitive.

Next suppose that G is irreducible but not primitive. Then for some $d>1$, we have $d|n$, say $n=dm$, where $V=V_1\oplus\cdots\oplus V_d$, the V_i spaces of imprimitivity of G. Let $G_i=\{g\in G|\ V_i^g=V_i\}$ and $N=\bigcap_{i=1}^d G_i$, so $N\lhd G$. By Theorem 3 of [5],

$$|G:N| \leqq 2^{d-1}3^{(d-1)/3}.$$

We now investigate the abelian group $Z(N)\lhd G$. We may assume each G_i irreducible and primitive on V_i; for if not, we could choose a larger d where the corresponding $\{G_i|_{V_i}\}$ were primitive. Denote

$$H_i = \{h\in G_i|\ hg|_{V_i} = gh|_{V_i},\text{ all } g\in G_i\}.$$

The primitive case above shows that

$$\begin{aligned}|G_i:H_i| &\leqq 2^{m-1}3^{(2m-1)/3}m^2 &&\text{if } m\neq 3,\\ &\leqq 216 &&\text{if } m=3.\end{aligned}$$

Of course $H_i\lhd G_i$, and $\bigcap_{i=1}^d H_i\subseteq Z(N)$. We have

$$|N:N\cap H_i| = |NH_i:H_i| \leqq |G_i:H_i|,$$

so

$$|N:Z(N)| \leqq \left|N:\bigcap_{i=1}^d H_i\right| = \left|N:\bigcap_{i=1}^d (N\cap H_i)\right| \leqq \prod_{i=1}^d |N:N\cap H_i|.$$

If $m=3$, this tells us

$$|G:Z(N)| \leqq 2^{d-1}3^{(d-1)/3}(216)^d = 2^{4n/3-1}3^{10n/9-1/3},$$

as desired.

If $m\neq 3$, then we get

$$\begin{aligned}|G:Z(N)| &\leqq 2^{d-1}3^{(d-1)/3}(2^{m-1}3^{(2m-1)/3}m^2)^d\\ &= 2^{n-1}3^{(2n-1)/3}m^{2d} = 2^{n-1}3^{(2n-1)/3}(m)^{2n/m}\\ &\leqq 2^{n-1}3^{(2n-1)/3}2^n = 2^{2n-1}3^{(2n-1)/3}\\ &< 2^{4n/3-1}3^{10n/9-1/3},\quad\text{as desired,}\end{aligned}$$

since $m^{1/m}\leqq\sqrt{2}$ for $m\geqq 4$.

Finally, suppose G is reducible on the vector space V, say $V=V_1\oplus\cdots\oplus V_k$, the V_i G-invariant and G-irreducible. Then by the previous cases there exist $A_i\lhd G$, $A_i|_{V_i}$ abelian, and if $n_i=\dim V_i$,

$$|G:A_i| \leqq 2^{4n_i/3-1}3^{10n_i/9-1/3},$$

by above. Hence $A=\bigcap_{i=1}^k A_i\lhd G$ is abelian, and

$$|G:A| \leqq \prod_{i=1}^{k} |G:A_i| \leqq 2^{4n/3-k}3^{10n/9-k/3}$$

$$\leqq 2^{4n/3-1}3^{10n/9-1/3}.$$

This completes the proof of the first part of our theorem. We now construct the examples required to prove the second part.

On p. 109 of [1], we see described a finite primitive solvable subgroup G^* of GL(3, **C**), called the Hessian group, with $|G^*: Z(G^*)| = 216$. We know from our earlier considerations that the Fitting subgroup of G^* has the form $P_0 \circ Z(G^*)$ (central product), where P_0 is extra special of order 27 and exponent 3. Choose a prime $p \nmid |G^*|$, and set $G_0 = G^* \times C$, C cyclic of order p; thus G_0 is also a primitive solvable complex linear group of degree 3, with $|G_0: Z(G_0)| = 216$ and $F(G_0) = P_0 \circ Z(G_0)$ (central product). Either G_0 or G^* serves to show our theorem gives the best possible bound for $n=3$.

If $k>0$, $n=3\cdot 4^k$, let N be the group of all diagonal block matrices diag$(x_1, \cdots, x_{4^k})$, each $x_i \in G_0$; thus $N \subseteq \mathrm{GL}(n, \mathbf{C})$, and $|N: Z(N)| = (216)^{4^k}$. By Theorem 3 of [5], there exists a solvable subgroup of the symmetric group on 4^k letters with order

$$2^{4^k-1}3^{(4^k-1)/3}.$$

Hence we may construct a solvable group H of permutation matrices in GL(n, **C**), made entirely of 3×3 blocks

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

H with the above order. H normalizes N, and $G_k = HN$ is a solvable subgroup of GL(n, **C**) with

$$|G_k:Z(N)| = 2^{4n/3-1}3^{10n/9-1/3},$$

our exact bound for $n = 3\cdot 4^k$.

To complete the proof, we shall show that every normal abelian subgroup A of G_k is contained in $A_k = Z(N)$. At least $A \subseteq F(G_k)$, the Fitting subgroup of G_k. We first show $F(G_k) \subseteq N$; consider any $x \in G_k - N$. Let B_0 be the Sylow p-subgroup of G_k, elementary abelian of order p^{4^k}, and let B be the subgroup generated by B_0 and x. Then clearly x does not centralize B_0, so B is not nilpotent and $B \not\subseteq F(G_k)$. But $B_0 \subseteq Z(N) \subseteq F(G_k)$, so $x \notin F(G_k)$, proving $F(G_k) \subseteq N$.

We now know $A \subseteq N$. Hence

$$A \subseteq F(G_0) \times \cdots \times F(G_0) \qquad (4^k \text{ factors}),$$

the Fitting subgroup of N. Each $F(G_0)=P_0 \circ Z(G_0)$ (central product) as we saw above. If there is some $x \in A-Z(N)$, say $x=x_1y_1 \cdots x_{4^k}y_{4^k}$, x_i in the ith factor P_0, y_i in the ith factor $Z(G_0)$, then we have some $x_j \in P_0-Z(P_0)$. We saw before that elements of G_0 perform on P_0 all possible automorphisms of P_0 fixing $Z(P_0)$ pointwise. Hence there is always an α in the jth factor G_0 such that $\langle x_j, x_j^\alpha\rangle = P_0$. We then have $x, x^\alpha \in A$, but

$$[x, x^\alpha] = [x_1y_1 \cdots x_jy_j \cdots x_{4k}y_{4k}, x_1y_1 \cdots x_j^\alpha y_j \cdots x_{4k}y_{4k}]$$
$$= [x_j, x_j^\alpha] \neq 1,$$

a contradiction to the fact that A is abelian.

We conclude that $A \subseteq Z(N)$, completing the proof of our theorem.

We conclude this paper with some remarks on the nonsolvable case. Our treatment in the key case (G primitive) does not need G solvable in the study of $|F(G):Z(G)|$. Hence bounds in Jordan's theorem could be improved if one could obtain better bounds on $|P:O_p(G)|$ (P a Sylow p-subgroup of G) than those obtained by Blichfeldt [1], especially for p relatively large when compared to n. (For $p>n+1$, this is settled in [6].) A tentative conjecture on $|P:O_p(G)|$ in general may be found at the end of [4].

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