GENERATORS AND RELATIONS FOR COXETER GROUPS

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The original proof of Coxeter's Theorem on generators and relations of reflection groups was topological in nature. Algebraic proofs were given in [1] and [3]. We present here an algebraic proof, shorter than the previous ones, having the advantage that it admits a geometric interpretation closely related to Coxeter's original proof.

COXETER'S THEOREM [2, p. 599]. *Suppose V is a real Euclidean vector space and G is a finite subgroup of the orthogonal group* $\mathcal{O}(V)$ *generated by reflections. Then G has a presentation*

$$G = \langle S_i \mid (S_iS_j)^{a_{ij}} = 1, \quad a_{ii} = 1, \quad 1 \leqq i \leqq j \leqq n \rangle.$$

We use the notation of [1]. The *length* $l(T)$ of an element T of G is the minimal number of factors possible when T is represented as a word in the fundamental reflections S_i. The words $S_{i_1} \cdots S_{i_j}$, with $0 \leqq j \leqq k$, are called *partial words* of the word $S_{i_1} \cdots S_{i_k}$. The symbol $(S_iS_j \cdots)_m$ will denote a word in alternating S_i's and S_j's, beginning with S_i and having m factors, $m \geqq 0$. Similar remarks apply to $(\cdots S_iS_j)_m$ and $(\cdots S_iS_j \cdots)_m$.

LEMMA. *Suppose* $S \in G$, *i and j are fixed, and* $l(SS_i) = l(SS_j) = l(S) - 1$. *Then* $l(S(\cdots S_iS_j \cdots)_m) = l(S) - m$ *if* $0 \leqq m \leqq a_{ij}$.

PROOF. The conclusion is trivial if $m = 0$, so suppose $m \geqq 1$ and that the result holds for $m - 1$. If a_i and a_j are the fundamental roots corresponding to S_i and S_j, then by Lemma 2.2 of [3] we have Sa_i, $Sa_j \in -\Sigma$. The reflections S_i and S_j generate a dihedral group of order $2a_{ij}$, and it is easy to see that $l((\cdots S_iS_j)_m) = m$. It follows that $(\cdots S_jS_i)_{m-1}a_j \in \Sigma$, again by Lemma 2.2 of [3], and so $(\cdots S_jS_i)_{m-1}a_j = \alpha a_i + \beta a_j$, with $\alpha, \beta \geqq 0$. Thus

$$S(\cdots S_jS_i)_{m-1}a_j = \alpha Sa_i + \beta Sa_j \in - \Sigma,$$

and

$$\begin{aligned} l(S(\cdots S_iS_j)_m) &= l(S(\cdots S_jS_i)_{m-1}) - 1 = l(S) - (m-1) - 1 \\ &= l(S) - m. \end{aligned}$$

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PROOF OF THE THEOREM. We shall show that every relation $W=S_{i_1}\cdots S_{i_k}=1$ is a consequence of the relations $(S_iS_j)^{a_{ij}}=1$. Suppose u is the maximal length of partial words of W. Then we may write W as $W_1S_iS_jW_2$, where $l(W_1S_i)=u$ and every partial word of W_1 has length less than u. Set $a=a_{ij}$, let $W'=W_1(S_jS_i\cdots)_{2a-2}W_2$, and observe that W and W' are equal as elements of G. With the exception of W_1S_i it is clear that all partial words of W are equal, as group elements, to partial words of W'. In place of W_1S_i, W' has the partial words W_1S_j, $W_1S_jS_i, \cdots$, $W_1(S_jS_i\cdots)_{2a-3}$. Setting $S=W_1S_i$, and using the fact that $S_i^2=1$, we see that the latter partial words coincide as group elements with $S(S_iS_j\cdots)_v$, $2\leqq v\leqq 2a-2$. Each of these has length less than $l(S)=u$. This is a direct consequence of the lemma if $v\leqq a$, otherwise it is a consequence of the lemma and the fact that $(S_iS_j\cdots)_v=(S_jS_i\cdots)_{2a-v}$, since then $2a-v<a$. Replacing W by W' we have removed the first partial word of maximal length in W. The procedure may be repeated as necessary until we arrive at the relation $1=1$, and the theorem is proved.

For the geometrical interpretation we associate with each word $S_{i_1}\cdots S_{i_k}$ a path in V (or in a simply connected subset invariant under G) as in [2, p. 600]. If F is the initial fundamental region, then the length of a word $S_{i_1}\cdots S_{i_k}$ is the minimal number of fundamental regions crossed by paths from F to $S_{i_1}\cdots S_{i_k}F$.

The steps of the proof above may be illustrated for the group G of symmetries of the cube in three dimensions. In this case

$$G = \langle S_1, S_2, S_3 \mid S_i^2 = (S_1S_2)^2 = (S_1S_3)^3 = (S_2S_3)^4 = 1\rangle.$$

As a simple example of a relation in G we have

$$W = S_3S_1S_2S_3S_1S_3S_1S_3S_2S_1S_2S_1S_2S_3 = 1.$$

The corresponding closed path is illustrated in the figure, with Roman numerals indicating lengths of partial words. Thus $S_3S_1S_2S_3S_1$ is the partial word of maximal length, and $l(S_3S_1S_2S_3S_1)=u=5$. Applying the relation $(S_1S_3)^3=1$ we obtain the relation

$$W' = S_3S_1S_2S_3S_3S_1S_3S_1S_1S_3S_2S_1S_2S_1S_2S_3 = 1$$

and the path with the dotted line. The new word W' has two partial words of length 4, viz. $S_3S_1S_2S_3$ and $S_3S_1S_2S_3S_3S_1S_3S_1$, but no partial words of length 5 or greater.

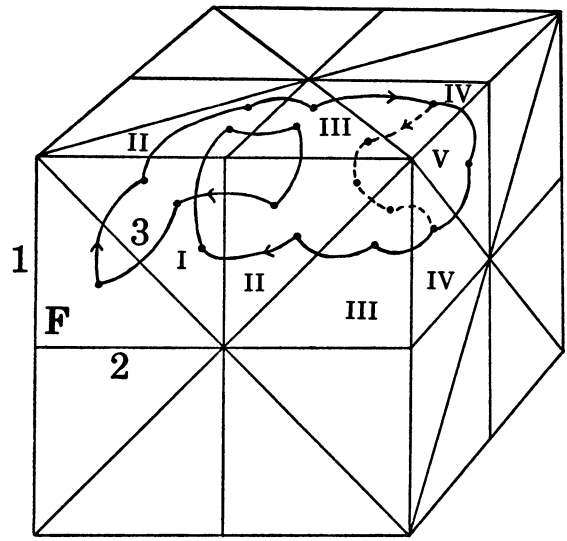

FIGURE 1

The process continues, giving the further sequence of relations

$$S_3S_1S_2S_1S_3S_1S_1S_3S_2S_1S_2S_1S_2S_3 = 1,$$
$$S_3S_1S_2S_1S_3S_3S_2S_1S_2S_1S_2S_3 = 1,$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$S_3S_3 = 1, \quad \text{and finally } 1 = 1.$$

As the example shows, the removal of a partial word of maximal length corresponds to Coxeter's device of shrinking the path through a wall (if $i=j$), or past an edge (if $i\neq j$), of a fundamental region.

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