A GENERAL METHOD FOR APPROXIMATING MEASURE PRESERVING TRANSFORMATIONS[1]

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1. Introduction. In references [1], [2], and [3] the method of approximation by periodic transformations has been used to investigate the properties of measure preserving transformations. In particular conditions have been found under which a transformation will be ergodic, weakly mixing or not strongly mixing. In this paper we define a generalization of the notion of approximation given by Katok and Stepin in [3] and show that several of the theorems given in that paper remain valid if our more general notion of approximation is employed. Since generalizations of the elegant proofs given in [3] cannot be used with the new definition, we have developed proofs depending on properties of the method of approximation which do not seem to have been utilized before and which are given in §3.

2. Preliminary Definitions. We will be concerned with transformations defined on the space $(X, \mathfrak{F}, \mu)$ where X is the unit interval, $\mathfrak{F}$ the Lebesgue measurable sets, and μ Lebesgue measure. An automorphism T is an invertible transformation of X onto X which is measure preserving; that is, $A \in \mathfrak{F}$ if and only if $TA \in \mathfrak{F}$, and $\mu(A) = \mu(TA) = \mu(T^{-1}A)$ for any $A \in \mathfrak{F}$. All automorphisms are understood to be defined modulo sets of measure zero, so that all statements hold almost everywhere.

DEFINITION 2.1. If $\xi = \{C_i, i = 1, \cdots, q\}$ is a collection of pairwise disjoint measurable sets whose union $X_\xi \subset X$, we say that ξ is a *partition*. If further, $X_\xi = X$ then we say that ξ is a *partition of X*.

If $A \in \mathfrak{F}$, then among the sets which are unions of elements of ξ there must be at least one set whose symmetric difference with A is minimal; we denote this set by $A(\xi)$.

DEFINITION 2.2. If $\xi(n) = \{C_i(n), i = 1, \cdots, q(n)\}$ is a sequence of partitions, we write $\xi(n) \to \epsilon$ as $n \to \infty$ if $\mu(A \Delta A(\xi(n))) \to 0$ as $n \to \infty$ for each set $A \in \mathfrak{F}$.

We give now the definition of cyclic approximation by periodic transformations given by Katok and Stepin in [3].

DEFINITION 2.3. Let $\{f(n)\}$ be a monotonic sequence of positive numbers such that $\lim_{n\to\infty} f(n) = 0$. We say that the automorphism T admits a cyclic approximation by periodic transformations with speed

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$f(n)$ if for each positive integer n there exists a partition of X, $\xi(n)=\{C_i(n), i=1, \cdots, q(n)\}$, and a measure preserving transformation T_n such that

1. $\xi(n)\to\epsilon$ as $n\to\infty$,
2. T_n maps the elements of $\xi(n)$ cyclically:

$$T_nC_i(n) = C_{i+1}(n), \quad i = 1, \cdots, q(n)-1, \; T_nC_{q(n)}(n) = C_i(n),$$

3. $\sum_{i=1}^{q(n)} \mu(TC_i(n)\Delta T_nC_i(n)) < f(q(n))$.

Next we state the theorems given by Katok and Stepin in [3] which will be generalized in §4.

THEOREM 2.1. *If the automorphism T admits a cyclic approximation by periodic transformations with speed θ/n, then the number of ergodic components of T does not exceed $\theta/2$.*

COROLLARY 2.1. *If the automorphism T admits a cyclic approximation by periodic transformations with speed θ/n with $\theta<4$ then T is ergodic.*

THEOREM 2.2. *If the automorphism T admits a cyclic approximation by periodic transformations with speed θ/n with $\theta<2$ then T is not strongly mixing.*

3. **Approximation without periodic transformations.** In Definition 2.3 we have assumed that T_n is measure preserving, whereas Katok and Stepin actually assume only that T_n preserves the measure of the elements of $\xi(n)$ so that $\mu(C_i(n))=1/q(n)$ for $i=1, \cdots, q(n)$. It is easy to see that our assumption involves no loss of generality and that we could just as well assume that T_n is an automorphism for each n. It is neither necessary nor advantageous to assume that T_n is periodic. In fact, the transformations T_n are extraneous to the definition, their only purpose being to insure that the elements of $\xi(n)$ will have equal measure. In our definition of approximation we will dispense with the transformations T_n and weaken the condition which they impose.

DEFINITION 3.1. Let $\{f(n)\}$ be a monotonic sequence of positive numbers such that $\lim_{n\to\infty} f(n)=0$. We say that the automorphism T admits an approximation with speed $f(n)$ if for each positive integer n there exists a partition $\xi(n)=\{C_i(n), i=1, \cdots, q(n)\}$ such that

1. $\xi(n)\to\epsilon$ as $n\to\infty$,
2. $\lim_{n\to\infty} \sum_{i=1}^{q(n)} |\mu(C_i(n)) - 1/q(n)| = 0$,
3. $\sum_{i=1}^{q(n)} \mu(TC_i(n)\cap C'_{i+1}(n)) < f(q(n))$,

where $C'_{i+1}(n)$ indicates the complement of the set $C_{i+1}(n)$ with respect to the whole space and where $C_{q(n)+1}(n)$ is understood to be $C_1(n)$.

REMARKS. It might seem more natural in the above definition to replace condition 3 with

$$\sum_{i=1}^{q(n)} \mu(TC_i(n)\Delta C_{i+1}(n)) < f(q(n)).$$

However since

$$\mu(C_{i+1}(n)) + \mu(TC_i(n) \cap C'_{i+1}(n)) = \mu(C_i) + \mu((TC_i(n))' \cap C_{i+1}(n)),$$

we have

$$\begin{aligned}\sum_{i=1}^{q(n)} \mu(TC_i(n) \cap C'_{i+1}(n) &= \sum_{i=1}^{q(n)} \mu((TC_i(n))' \cap C_{i+1}(n)) \\ &= \frac{1}{2}\sum_{i=1}^{q(n)} \mu(TC_i(n)\Delta C_{i+1}(n)).\end{aligned}$$

We note also that we have not assumed $\bigcup_{i=1}^{q(n)} C_i(n) = X$, but condition 1 implies that $\lim_{n\to\infty} \mu(\bigcup_{i=1}^{q(n)} C_i(n)) = 1$.

The following three lemmas establish properties of a sequence of partitions which satisfy the conditions of Definition 3.1. It is these properties which are utilized in the proofs of the main theorems in §4.

LEMMA 3.1. *Let* $\xi(n) = \{C_i(n),\ i=1, \cdots, q(n)\}$ *be a sequence of partitions such that* $\lim_{n\to\infty} \sum_{i=1}^{q(n)} |\mu(C_i(n)) - 1/q(n)| = 0$. *Fix a number* η, $0<\eta<1$, *and define the set*

$$I_\eta(n) = \left\{i, \left|\mu(C_i(n)) - \frac{1}{q(n)}\right| > \frac{\eta}{q(n)}\right\}$$

then $\lim_{n\to\infty} \sum_{i\in I_\eta(n)} \mu(C_i(n)) = 0$.

PROOF. Let $N(I_\eta(n))$ denote the number of elements in $I_\eta(n)$. It then follows that

$$\eta\,\frac{N(I_\eta(n))}{q(n)} \leqq \sum_{i=1}^{q(n)} \left|\mu(C_i(n)) - \frac{1}{q(n)}\right|,$$

so that

$$\begin{aligned}\sum_{i\in I_\eta(n)} \mu(C_i(n)) &= \sum_{i\in I_\eta(n)} \frac{1}{q(n)} + \sum_{i\in I_\eta(n)} \left(\mu(C_i(n)) - \frac{1}{q(n)}\right) \\ &\leqq \frac{N(I_\eta(n))}{q(n)} + \sum_{i=1}^{q(n)} \left|\mu(C_i(n)) - \frac{1}{q(n)}\right| \\ &\leqq \left(\frac{1}{\eta} + 1\right) \sum_{i=1}^{q(n)} \left|\mu(C_i(n)) - \frac{1}{q(n)}\right|.\end{aligned}$$

LEMMA 3.2. *Let* $\xi(n)=\{C_i(n),\ i=1,\cdots,q(n)\}$ *be a sequence of partitions such that* $\xi(n)\to\epsilon$ *as* $n\to\infty$. *Fix a number* η, $0<\eta<1$, *and define for* $E\in\mathfrak{F}$

$$I_{\eta,E}(n) = \{i \mid \mu(E\cap C_i(n)) < (1-\eta)\mu(C_i(n))\}.$$

Let $G(n)$ *be the set determined by*

$$E(\xi(n)) = \sum_{i\in G(n)} C_i(n),$$

then $\lim_{n\to\infty}\sum_{i\in G(n)\cap I_{\eta,E}(n)}\mu(C_i(n))=0$.

PROOF. Since

$$\mu(E\Delta E(\xi(n))) \geqq \sum_{i\in G(n)\cap I_{\eta,E}(n)} \eta\mu(C_i(n)),$$

it follows that

$$\lim_{n\to\infty}\sum_{i\in G(n)\cap I_{\eta,E}(n)}\mu(C_i(n)) \leqq \frac{1}{\eta}\lim_{n\to\infty}\mu(E\Delta E(\xi(n))) = 0.$$

In the following lemma $G(n)$, $I_\eta(n)$, and $I_{\eta,E}(n)$ refer to the sets defined in the preceding two lemmas.

LEMMA 3.3. *Let* $\xi(n)=\{C_i(n),\ i=1,2,\cdots,q(n)\}$ *be a sequence of partitions such that* $\xi(n)\to\epsilon$ *as* $n\to\infty$ *and*

$$\lim_{n\to\infty}\sum_{i=1}^{q(n)}\left|\mu(C_i(n)) - \frac{1}{q(n)}\right| = 0.$$

Then if $E\in\mathfrak{F}$ *there exists a sequence* $\{\eta_n\}$ *of positive numbers such that* $\lim_{n\to\infty}\eta_n=0$ *and such that if* $H(n)=G(n)\cap I'_{\eta_n}(n)\cap I'_{\eta_n,E}(n)$ *then* $\lim_{n\to\infty}N(H(N))/q(n)=\mu(E)$ *where* $N(H(n))$ *denotes the number of elements in* $H(n)$.

PROOF. If the sequence $\{\eta_n\}$ is chosen properly we can assume, by choosing a subsequence of the partitions $\xi(n)$ if necessary, that

$$\lim_{n\to\infty}\sum_{i\in I_{\eta_n}(n)}\mu(C_i(n)) = \lim_{n\to\infty}\sum_{i\in I_{\eta_n,E}(n)}\mu(C_i(n)) = 0$$

so that

$$\sum_{i\in H(n)}\mu(E\cap C_i(n)) \leqq \mu(E) \leqq \sum_{i\in H(n)}\mu(C_i(n)) + \sum_{i\in H'(n)}\mu(E\cap C_i(n))$$

which implies that

$$\limsup_{n\to\infty} N(H(n))\frac{(1-\eta_n)^2}{q(n)} \leqq \mu(E) \leqq \liminf_{n\to\infty} N(H(n))\frac{1+\eta_n}{q(n)}$$

which gives us

$$\lim_{n\to\infty} \frac{N(H(n))}{q(n)} = \mu(E).$$

4. Principal results. In this section we obtain results that are generalizations of Theorems 2.1 and 2.2 by replacing approximation by periodic transformations with the type of approximation given in Definition 3.1.

THEOREM 4.1. *If the automorphism T admits an approximation with speed θ/n then the number of ergodic components of T is not greater than θ.*

PROOF. Let $E_1, E_2, \cdots, E_M$ be pairwise disjoint sets which are invariant under T. If we fix a number η, $0<\eta<1$ then by Lemmas 3.1 and 3.2, for n sufficiently large we can assume that for each invariant set E_j there is an element $C_{i(j)}(n)$ of $\xi(n)$ such that

$$\left|\mu C_{i(j)}(n) - \frac{1}{q(n)}\right| < \frac{\eta}{q(n)}$$

and $\mu(E_j \cap C_{i(j)}(n)) > (1-\eta)\mu(C_{i(j)}(n))$. We may also assume that the sets $E_1, E_2, \cdots, E_M$ are arranged in such an order that $i(j)<i(j+1)$ for $j=1, 2, \cdots, M-1$ and that $i(M+1)=i(1)$. Let $F_j(n)$ be the largest subset of $E_j \cap C_{i(j)}(n)$ such that $T^k F_j(n)$ does not intersect $\sum_{i=i(j)}^{i(j+1)-1} TC_i(n) \cap C'_{i+1}(n)$ for $k=1, 2, \cdots, i(j+1)-i(j)-1$. Then since E_j is invariant $T^{i(j+1)-i(j)-1}F_j(n) \subset E_j$. We have also that $T^{i(j+1)-i(j)-1}F_j(n) \subset C_{i(j+1)}(n)$ and $\mu(E_j \cap C_{i(j+1)}(n)) < \eta\mu(C_{i(j+1)}(n)) < \eta(1+\eta)/q(n)$. Since T is measure preserving it follows that $\mu(F_j) < \eta(1+\eta)/q(n)$ so that

$$\sum_{i=i(j)}^{i(j+1)-1} TC_i(n) \cap C'_{i+1}(n) \geqq \mu(E_j \cap C_{i(j)}(n)) - \mu(F_j)$$

$$\geqq \frac{(1-\eta)^2}{q(n)} - \frac{\eta+\eta^2}{q(n)} = \frac{1-3\eta}{q(n)}.$$

Therefore

$$\frac{M(1-3\eta)}{q(n)} \leqq \sum_{i=1}^{q(n)} \mu(TC_i(n) \cap C'_{i+1}(n)) < \frac{\theta}{q(n)}.$$

Since η is arbitrary this implies that $M \leqq \theta$.

COROLLARY 4.1. *If the automorphism T admits an approximation with speed θ/n with $\theta<2$ then T is ergodic.*

THEOREM 4.2. *If the automorphism T admits an approximation with speed θ/n with $\theta<1$ then T is not strongly mixing.*

PROOF. Let $E\in\mathfrak{F}$ and let $\{\eta_n\}$ be the sequence of numbers and $H(n)$ the sets defined in Lemma 3.3. Since $\lim_{n\to\infty}\eta_n=0$ we can assume that there exists a number η such that $\eta_n<\eta<1-\theta$ for all n. If $i\in H(n)$ then

$$\left|\mu(C_i(n))-1/q(n)\right| < \eta_n/q(n) \quad\text{and}\quad \mu(E\cap C_i(n)) > (1-\eta_n)\mu(C_i(n)).$$

Since $\sum_{i=1}^{q(n)} T^iC_i(n)\cap C'_{i+1}(n)<\theta/q(n)$ for $i\in H(n)$ there exists a set $F_i(n)\subset C_i(n)$ such that $T^kF_i(n)$ does not intersect $\sum_{i=1}^{q(n)} TC_i(n)\cap C'_{i+1}(n)$ for $k=1, 2, \cdots, q(n)$, so that $T^{q(n)}F_i(n)\subset C_i(n)$, and

$$\mu(F_i(n)) \geqq \mu(C_i(n)) - \frac{\theta}{q(n)} \geqq \frac{1-\eta_n}{q(n)} + \frac{\eta-1}{q(n)} = \frac{\eta-\eta_n}{q(n)}.$$

If we let $E(n)=\sum_{i\in H(n)} C_i(n)$ then Lemmas 3.1, 3.2, and 3.3 imply that $\lim_{n\to\infty}\mu(E\Delta E(n))=0$ so that

$$\begin{aligned}\lim_{n\to\infty}\mu(T^{q(n)}E\cap E) &= \lim_{n\to\infty}\mu(T^{q(n)}E(n)\cap E(n))\\ &\geqq \lim_{n\to\infty} N(H(n))\frac{(\eta-\eta_n)}{q(n)} = \eta\mu(E).\end{aligned}$$

This contradicts strong mixing if we choose $E\in\mathfrak{F}$ such that $\mu(E)<\eta$.

REFERENCES

1. R. V. Chacon, *Weakly mixing transformations which are not strongly mixing*, Proc. Amer. Math. Soc. 22 (1969), 559–562.

2, ———, *Approximation of transformations with continuous spectrum*, Pacific J. Math. (to appear).

3. A. B. Katok and A. M. Stepin, *Approximations in ergodic theory*, Uspehi Mat. Nauk 22 (1967), no. 5 (137), 81–106 = Russian Math. Surveys 22 (1967), no. 5, 77–102. MR 36 #2776.

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