THE PEAK SETS OF $A^m$

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Abstract. Let $A$ denote the algebra of functions analytic for $|z| < 1$ and continuous for $|z| \leq 1$. For $m = 1, 2, \ldots$, let $A^m$ be the algebra of functions $f$ such that $f, f', \ldots, f^{(m)} \in A$; and let $A^n = \bigcap_{m=1}^n A^m$. We show that the peak sets of $A^m$, $1 \leq m \leq \infty$, are the finite subsets of $\{|z| = 1\}$.

Recall that a nonempty proper closed subset $E$ of $\{|z| \leq 1\}$ is a peak set for $A^m (A)$ iff there exists $f \in A^m (A)$ with $f(z) = 1$ for $z \in E$ and $|f(z)| < 1$ for $z \in E$. By the maximum principle, a peak set for $A^m (A)$ must be a subset of $\{|z| = 1\}$.

Rudin [4], [2, pp. 80–82] has shown that the peak sets for $A$ are the closed subsets of $\{|z| = 1\}$ with Lebesgue measure zero. In his paper [3] on the boundary zeros of $A^\infty$, Novinger raised the question of determining the peak sets of $A^\infty$. It is interesting that the boundary zero sets for $A^m$, $1 \leq m \leq \infty$, are the Carleson sets [1], [3], [5] while the peak sets for $A^m$ are much more restricted.

Theorem 1. A nonempty closed subset $E$ of $\{|z| = 1\}$ is a peak set for $A^m (A)$, $1 \leq m \leq \infty$, iff $E$ is finite.

Theorem 1 is a consequence of the following two theorems.

Here, for $f \in A$,

$$||f||_\infty = \sup \{|f(z)| : |z| = 1\}.$$ 

Theorem 2. Let $E$ be a nonempty closed subset of $\{|z| = 1\}$. If there exists a nonconstant function $f \in A^1$ such that $f(z) = 1$ for $z \in E$ and $||f||_\infty = 1$, then $E$ is finite.

Proof. Suppose $E$ is not finite. Then, without loss of generality, we may assume that $1$ is a limit point of $E$. Now $g = 1 - f \in A^1$, $g$ is not identically zero, and $g$ maps $\{|z| < 1\}$ into $\{|z - 1| < 1\}$. Thus $|\arg g(z)| < \pi/2$ for $|z| < 1$. Since $-\log |g|$ is the harmonic conjugate of $\arg g$ in $\{|z| < 1\}$,
\[-\log |g(re^{i\theta})| = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{2r \sin(\theta - t)}{1 - 2r \cos(\theta - t) + r^2} \arg g(e^{it}) \, dt;\]

and so for $0 \leq r < 1$,

\[|\log |g(re^{i\theta})|| \leq \frac{1}{2} \int_{-\pi}^{\pi} \frac{r |\sin(\theta - t)|}{1 - 2r \cos(\theta - t) + r^2} \, dt\]

\[= \int_{0}^{\pi} \frac{r \sin t \, dt}{1 - 2r \cos t + r^2} = \log \frac{1 + r}{1 - r}.\]

Hence $|g(r)| \geq \frac{1}{2}(1 - r)$ for $0 < r < 1$. However, since $g$ vanishes on $E$ and $1$ is a limit point of $E$, $g(1) = g'(1) = 0$. Therefore $g(r) = o(1-r)$ as $r \to 1^-$, which is a contradiction.

That the peak sets of $A^m$, $1 \leq m \leq \infty$, are finite subsets of $\{ |z| = 1 \}$ is immediate from Theorem 2. The case $\beta_i = 1$, $i = 1, \ldots, n$, in the following elementary interpolation theorem shows that every finite subset of $\{ |z| = 1 \}$ is a peak set for $A^m$, $1 \leq m \leq \infty$.

**Theorem 3.** Let $\alpha_1, \ldots, \alpha_n$ be distinct complex numbers with $|\alpha_i| = 1$, $i = 1, \ldots, n$; and let $\beta_1, \ldots, \beta_n$ be complex numbers with $M = \sup \{|\beta_1|, \ldots, |\beta_n|\} > 0$. There exists a rational function $f$ with all poles in $\{ |z| > 1 \}$ such that $f(\alpha_i) = \beta_i$, $i = 1, \ldots, n$, and $|f(z)| < M$ for $|z| \leq 1$, $z \neq \alpha_i$, $i = 1, \ldots, n$.

**Proof.** We can assume without loss of generality that $\beta_i \neq -1$, $i = 1, \ldots, n$, and that $M = 1$. Using the map $k(z) = (z-1)/(z+1)$ of $\{ |z| \leq 1 \}$ onto $\{ \text{Re } k(z) \leq 0 \}$, we can obtain the desired function $f$ in the form $f = k^{-1} \circ g$ where $g$ is a rational function with all poles in $\{ |z| > 1 \}$ such that

$g(\alpha_i) = \gamma_i = (\beta_i - 1)/(\beta_i + 1), \quad i = 1, \ldots, n,$

and $\text{Re } (\alpha_i - 1)^{-1} \leq -\frac{1}{2}$ for $|z| \leq 1$, $z \neq \alpha_i$, $i = 1, \ldots, n$.

When $n = 1$, we can take $g(z) = \bar{\alpha}_1 z - 1 + \gamma_1$. When $n > 1$, it suffices to construct $g$ in the special case $\gamma_1 \neq 0, \gamma_i = 0$, $i = 2, \ldots, n$. For then, the solution in the general case is obtained by adding solutions of the special case. For each $\alpha_i$, $i = 2, \ldots, n$,

$\text{Re } (\bar{\alpha}_i - 1)^{-1} \leq -\frac{1}{2}$

for $|z| \leq 1$, $z \neq \alpha_i$. The rational function

$k(z) = \bar{\alpha}_1 z - 1 + \sum_{i=2}^{n} (\bar{\alpha}_i z - 1)^{-1} + \frac{n-1}{2} - i \text{ Im } \sum_{i=2}^{n} (\bar{\alpha}_i \alpha_1 - 1)^{-1} + \gamma_1^{-1}$

has the following properties:
(i) \( \|h(\alpha_i)\| = 1 \) and so \( \text{Re}(\bar{\alpha}_i \alpha_i - 1) = -\frac{1}{2} \),
(ii) \( h \) has poles only at \( \alpha_i, \ i = 2, \ldots, n \),
(iii) \( \text{Re} \ h(z) < 0 \) for \( \|z\| \leq 1, \ z \neq \alpha_i, \ i = 1, \ldots, n \).

Therefore the rational function \( g = 1/h \) is a solution to the special case of the interpolation problem.

References


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