# THE PEAK SETS OF $A^m$

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ABSTRACT. Let $A$ denote the algebra of functions analytic for $|z|<1$ and continuous for $|z|\leqq 1$. For $m=1, 2, \cdots$, let $A^m$ be the algebra of functions $f$ such that $f, f', \cdots, f^{(m)}\in A$; and let $A^\infty=\bigcap_{m=1}^\infty A^m$. We show that the peak sets of $A^m$, $1\leqq m\leqq\infty$, are the finite subsets of $\{|z|=1\}$.

Recall that a nonempty proper closed subset $E$ of $\{|z|\leqq 1\}$ is a peak set for $A^m$ ($A$) iff there exists $f\in A^m$ ($A$) with $f(z)=1$ for $z\in E$ and $|f(z)|<1$ for $z\notin E$. By the maximum principle, a peak set for $A^m$ ($A$) must be a subset of $\{|z|=1\}$.

Rudin [4], [2, pp. 80–82] has shown that the peak sets for $A$ are the closed subsets of $\{|z|=1\}$ with Lebesgue measure zero. In his paper [3] on the boundary zeros of $A^\infty$, Novinger raised the question of determining the peak sets of $A^\infty$. It is interesting that the boundary zero sets for $A^m$, $1\leqq m\leqq\infty$, are the Carleson sets [1], [3], [5] while the peak sets for $A^m$ are much more restricted.

THEOREM 1. *A nonempty closed subset $E$ of $\{|z|=1\}$ is a peak set for $A^m$, $1\leqq m\leqq\infty$, iff $E$ is finite.*

Theorem 1 is a consequence of the following two theorems.

Here, for $f\in A$,

$$\|f\|_\infty = \sup\{|f(z)| : |z| = 1\}.$$

THEOREM 2. *Let $E$ be a nonempty closed subset of $\{|z|=1\}$. If there exists a nonconstant function $f\in A^1$ such that $f(z)=1$ for $z\in E$ and $\|f\|_\infty=1$, then $E$ is finite.*

PROOF. Suppose $E$ is not finite. Then, without loss of generality, we may assume that 1 is a limit point of $E$. Now $g=1-f\in A^1$, $g$ is not identically zero, and $g$ maps $\{|z|<1\}$ into $\{|z-1|<1\}$. Thus $|\arg g(z)|<\pi/2$ for $|z|<1$. Since $-\log|g|$ is the harmonic conjugate of $\arg g$ in $\{|z|<1\}$,

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$$-\log|g(re^{i\theta})| = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{2r\sin(\theta-t)}{1-2r\cos(\theta-t)+r^2}\arg g(e^{it})dt;$$

and so for $0\leqq r<1$,

$$|\log|g(re^{i\theta})|| \leqq \frac{1}{2}\int_{-\pi}^{\pi}\frac{r|\sin(\theta-t)|}{1-2r\cos(\theta-t)+r^2}dt$$

$$= \int_{0}^{\pi}\frac{r\sin t dt}{1-2r\cos t+r^2} = \log\frac{1+r}{1-r}\cdot$$

Hence $|g(r)|\geqq\frac{1}{2}(1-r)$ for $0<r<1$. However, since $g$ vanishes on $E$ and 1 is a limit point of $E$, $g(1)=g'(1)=0$. Therefore $g(r)=o(1-r)$ as $r\to 1^-$, which is a contradiction.

That the peak sets of $A^m$, $1\leqq m\leqq\infty$, are finite subsets of $\{|z|=1\}$ is immediate from Theorem 2. The case $\beta_i=1$, $i=1,\cdots,n$, in the following elementary interpolation theorem shows that every finite subset of $\{|z|=1\}$ is a peak set for $A^m$, $1\leqq m\leqq\infty$.

THEOREM 3. *Let $\alpha_1,\cdots,\alpha_n$ be distinct complex numbers with $|\alpha_i|=1$, $i=1,\cdots,n$; and let $\beta_1,\cdots,\beta_n$ be complex numbers with $M=\sup\{|\beta_1|,\cdots,|\beta_n|\}>0$. There exists a rational function $f$ with all poles in $\{|z|>1\}$ such that $f(\alpha_i)=\beta_i$, $i=1,\cdots,n$, and $|f(z)|<M$ for $|z|\leqq 1$, $z\neq\alpha_i$, $i=1,\cdots,n$.*

PROOF. We can assume without loss of generality that $\beta_i\neq -1$, $i=1,\cdots,n$, and that $M=1$. Using the map $k(z)=(z-1)/(z+1)$ of $\{|z|\leqq 1\}$ onto $\{\text{Re } k(z)\leqq 0\}$, we can obtain the desired function $f$ in the form $f=k^{-1}\circ g$ where $g$ is a rational function with all poles in $\{|z|>1\}$ such that

$$g(\alpha_i) = \gamma_i = (\beta_i-1)/(\beta_i+1),$$

$i=1,\cdots,n$, and $\text{Re } g(z)<0$ for $|z|\leqq 1$, $z\neq\alpha_i$, $i=1,\cdots,n$.

When $n=1$, we can take $g(z)=\bar{\alpha}_1 z-1+\gamma_1$. When $n>1$, it suffices to construct $g$ in the special case $\gamma_1\neq 0$, $\gamma_i=0$, $i=2,\cdots,n$. For then, the solution in the general case is obtained by adding solutions of the special case. For each $\alpha_i$, $i=2,\cdots,n$,

$$\text{Re}(\bar{\alpha}_i z-1)^{-1}\leqq -\tfrac{1}{2}$$

for $|z|\leqq 1$, $z\neq\alpha_i$. The rational function

$$h(z) = \bar{\alpha}_1 z - 1 + \sum_{i=2}^{n}(\bar{\alpha}_i z-1)^{-1} + \frac{n-1}{2} - i\,\text{Im}\sum_{i=2}^{n}(\bar{\alpha}_i\alpha_1-1)^{-1} + \gamma_1^{-1}$$

has the following properties:

(i) $h(\alpha_1)=\gamma_1^{-1}$, since $|\bar{\alpha}_i\alpha_1|=1$ and so $\mathrm{Re}(\bar{\alpha}_i\alpha_1-1)^{-1}=-\frac{1}{2}$,

(ii) $h$ has poles only at $\alpha_i$, $i=2, \cdots, n$,

(iii) Re $h(z)<0$ for $|z|\leqq 1$, $z\neq\alpha_i$, $i=1, \cdots, n$.

Therefore the rational function $g=1/h$ is a solution to the special case of the interpolation problem.

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