NOTE ON THE HOMOLOGY OF A FIBER PRODUCT OF GROUPS

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ABSTRACT. Spectral sequences are derived for the homology and cohomology of a fiber product of groups with coefficients in a tensor product module. These generalize the Hochschild-Serre spectral sequences, and, in the case of a full product of groups, give Künneth formulas. The latter are used to make easy explicit computations of the homology and cohomology of an arbitrary finitely generated abelian group acting trivially on an arbitrary module.

We will prove the following:

THEOREM 1. *For* $i=1, 2$, *let* $f_i\colon B_i \to A$ *be a surjection of groups with kernel* K_i, *and let* M_i *be a left* B_i*-module. Let* $B_1 \times_A B_2 = \{(b_1, b_2) \in B_1 \times B_2 \mid f_1(b_1) = f_2(b_2)\}$, *and consider* $M_1 \otimes_Z M_2$ *as a left* $B_1 \times_A B_2$*-module in the natural fashion. If* N, N' *are left* A*-modules, let* ZA *denote the integral group ring of* A, *and form* $\mathrm{Tor}^{ZA}(N, N')$ *by considering* N *as a right* A*-module with* $na = a^{-1}n$. *Then, if* $\mathrm{Tor}_1^Z(M_1, M_2) = 0$, *there is a spectral sequence*

$$E^2_{p,q} \Rightarrow H_n(B_1 \times_A B_2, M_1 \otimes_Z M_2) \tag{1.1}$$

where

$$E^2_{p,q} = \sum_{s+t=q} \mathrm{Tor}_p^{ZA}(H_s(K_1, M_1), H_t(K_2, M_2)).$$

Again, consider $\mathrm{Hom}_Z(M_1, M_2)$ as a left $B_1 \times_A B_2$-module with $((b_1, b_2)f)(m) = b_2 f(b_1^{-1}m)$. Then, if $\mathrm{Ext}_Z^1(M_1, M_2) = 0$, there is a spectral sequence

$$\sum_{s+t=q} \mathrm{Ext}^p_{ZA}(H_s(K_1, M_1), H^t(K_2, M_2)) \Rightarrow H^n(B_1 \times_A B_2, \mathrm{Hom}_Z(M_1, M_2)). \tag{1.2}$$

COROLLARY. *If* $\mathrm{Tor}_1^Z(M_1, M_2) = 0$, *there are* Z*-split exact sequences*

$$\begin{aligned} 0 \to \sum_{s+t=n} H_s(B_1, M_1) \otimes_Z H_t(B_2, M_2) &\to H_n(B_1 \times B_2, M_1 \otimes_Z M_2) \\ \to \sum_{s+t=n-1} \mathrm{Tor}_1^Z(H_s(B_1, M_1), H_t(B_2, M_2)) &\to 0, \end{aligned} \tag{1.3}$$

and, if $\mathrm{Ext}_Z^1(M_1, M_2) = 0$,

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$$0 \to \sum_{s+t=n-1} \mathrm{Ext}_Z^1(H_s(B_1, M_1), H^t(B_2, M_2)) \to H^n(B_1 \times B_2, M_1 \otimes_Z M_2)$$
$$(1.4) \qquad \to \sum_{s+t=n} \mathrm{Hom}_Z(H_s(B_1, M_1), H^t(B_2, M_2)) \to 0.$$

Note that, for $B_1 = A$, $M_1 = Z$, (1.1) and (1.2) become the Hochschild-Lyndon-Serre spectral sequences. If $M_1 = Z$, $B_2 = \{e\}$, (1.3) and (1.4) give the Universal Coefficient Theorems. When B_i acts trivially on M_i, (1.3) follows from the Künneth formula and the Eilenberg-Zilber theorem in algebraic topology.

The sequences (1.3) and (1.4) give very easy explicit computations of the homology and cohomology of an arbitrary finitely generated abelian group acting trivially on an arbitrary module (cf. [3], [4]), and we close with these.

For the proof of Theorem 1, we will need the following generalizations to bifunctors of the spectral sequence associated to the composition of two functors.

THEOREM 2. *For $i = 1, 2$, let F_i: $\mathbf{C}_i \to \mathbf{B}_i$ be additive functors between abelian categories with enough projectives. Let $\mathbf{A}$ be an abelian category, and let G: $\mathbf{B}_1 \times \mathbf{B}_2 \to \mathbf{A}$ be additive in each variable. Suppose $L_pG(F_1(P_1), F_2(P_2)) = 0$ whenever P_1, P_2 are projective and $p \geqq 1$. Then there is a spectral sequence*

$$(2.1) \qquad \sum_{s+t=q} L_pG(L_sF_1(M_1), L_tF_2(M_2)) \Rightarrow L_n(G(F_1 \times F_2))(M_1, M_2).$$

Again, suppose that $\mathbf{A}$ has enough projectives, that $\mathbf{C}$ is abelian, and that F: $\mathbf{A} \to \mathbf{C}$ is additive. Then, if $L_pF(G(P_1, P_2)) = 0$ whenever P_1, P_2 are projectives and $p \geqq 1$, there is a spectral sequence

$$(2.2) \qquad L_pF(L_qG(N_1, N_2)) \Rightarrow L_n(FG)(N_1, N_2).$$

PROOF. The proof proceeds as in [2, Theorem 2.4.1]. Let us first consider, in the category $\mathbf{B}_1$, a complex U with the property that $B_n(U)$ and $Z_n(U)$ are direct summands of U_n, so that U has the form:

$$\begin{array}{l} \quad\vdots \\ \quad\downarrow \\ U_n \cong B_n(U) \oplus H_n(U) \oplus B_{n-1}(U) \\ \qquad\qquad\qquad\qquad\qquad \downarrow \\ \qquad\qquad\qquad U_{n-1} \cong B_{n-1}(U) \oplus H_{n-1}(U) \oplus B_{n-2}(U) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \downarrow \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \vdots \end{array}$$

Let U' be another such complex in $\boldsymbol{B}_2$. Consider $G(U, U')$, the complex associated to the double complex $\{G(U_p, U'_q)\}$. This decomposes into the direct sum of nine complexes because of the decomposition of U and U' into three as above. $H_nG(H(U), H(U')) = \sum_{s+t=n} G(H_s(U), H_t(U'))$, and the other eight factors have trivial homology. Hence

$$(2.3) \qquad H_nG(U, U') = \sum_{s+t=n} G(H_s(U), H_t(U')).$$

Now let X be a projective resolution of M_1, and construct, as in [1, Chapter XVII], a double complex Y such that, for each p, q, $B_q(Y_{p*})$ and $Z_q(Y_{p*})$ are direct summands of Y_{pq}, and, for each q, Y_{*q} and $H_q^{II}(Y_{**})$ are projective resolutions of $F_1(X_q)$ and $H_qF_1(X) = L_qF_1(M_1)$ respectively. Construct X', Y' for M_2 similarly. Let $V_{pq} = \sum_{s+t=p;s'+t'=q} G(Y_{ss'}, Y'_{tt'})$. Then

$$H_p(V_{*q}) = \sum_{s'+t'=q} L_pG(F_1(X_{s'}), F_2(X'_{t'})) = 0 \qquad (p \geqq 1)$$

and, by (2.3),

$$H_q(V_{p*}) = \sum_{s+t=p;s'+t'=q} G(H_{s'}(Y_{s*}), H_{t'}(Y_{t*})).$$

Hence one of the two spectral sequences associated to the double complex V collapses to yield $H_n(V) = L_n(G(F_1 \times F_2))(M_1, M_2)$, and the other thus becomes (2.1).

To establish (2.2), let $X^{(i)}$ be a projective resolution of N_i, and let Y be a double complex resolution (in the above sense) for the single complex associated to $\{G(X_p^{(1)}, X_q^{(2)})\}$.

PROOF OF THEOREM 1. For any group C, $H_n(C, -)$ is the nth left derived functor of the functor $M \mapsto M_C$, where M_C is the quotient of M by the submodule generated by $\{\sigma m - m \mid m \in M, \sigma \in C\} \odot (ZB_i)_{K_i} = ZA$, so that $M \mapsto M_{K_i}$ preserves projectives when considered as a functor to A-modules. Again, $ZB_1 \otimes ZB_2 = Z(B_1 \times B_2)$, which is projective for the subgroup $B_1 \times_A B_2 \subset B_1 \times B_2$ (consider a coset decomposition). Hence, since $M_{K_1} \otimes_A N_{K_2} = (M \otimes_Z N)_{B_1 \times_A B_2}$, we obtain from Theorem 2 two spectral sequences with the same limit. The second of these collapses, because $\mathrm{Tor}_1^Z(M_1, M_2) = 0$, to identify the limit. The first thus becomes (1.1).

$H^n(C, -)$ is the right derived functor of $M \mapsto M^C = \{m \in M \mid \sigma m = m$ for every $\sigma \in C\}$. Let I be an injective Z-module. The right C-module structure of ZC defines a left C-module structure for $\mathrm{Hom}_Z(ZC, I)$. The latter is injective, and every injective is a direct summand of one of this form. $(\mathrm{Hom}_Z(ZB_i, I))^{K_i} \cong \mathrm{Hom}_Z(ZA, I)$, so that $M \mapsto M^{K_i}$

preserves injectives. $\mathrm{Hom}_{ZA}(M_{K_1}, N^{K_2}) = (\mathrm{Hom}_Z(M, N))^{B_1 \times_A B_2}$. Hence the result for cohomology follows analogously.

PROOF OF COROLLARY. When $A = \{e\}$ the spectral sequences (1.1) and (1.2) become the desired short exact sequences. It remains to show that the latter split. For $n=0$, this is trivial. Let $0 \to M^1 \to P \to M_1 \to 0$ be exact with P a projective ZB_1-module. We obtain isomorphisms $H_n(B_1, M_1) \simeq H_{n-1}(B_1, M^1)$ $(n \geqq 2)$ and an exact sequence

$$0 \to H_1(B_1, M_1) \underset{\beta}{\to} H_0(B_1, M^1) \underset{\alpha}{\to} H_0(B_1, P).$$

Now $H_0(B_1, P) = P_{B_1}$ is Z-projective. Hence so is image α. Hence β is split. From the exact sequence $0 = \mathrm{Tor}_1^Z(M_1, M_2) \to M^1 \otimes M_2 \to P \otimes M_2 \to M_1 \otimes M_2 \to 0$ we also have a homomorphism $H_n(B_1 \times B_2, M_1 \otimes M_2) \to H_{n-1}(B_1 \times B_2, M^1 \otimes M_2)$. The diagram

$$\begin{array}{ccc} \sum_{l=0}^{n} H_{n-l}(B_1, M_1) \otimes H_l(B_2, M_2) & \underset{\gamma}{\to} & H_n(B_1 \times B_2, M_1 \otimes M_2) \\ \downarrow & & \downarrow \\ \sum_{l=0}^{n-1} H_{n-l-1}(B_1, M^1) \otimes H_l(B_2, M_2) & \underset{\gamma'}{\to} & H_{n-1}(B_1 \times B_2, M^1 \otimes M_2) \end{array}$$

commutes up to sign. (In fact, it comes from a morphism of double complexes of degree $(0, -1)$: In the proof of Theorem 2, choose X such that $X_0 = P$, and let $X''_q = X_{q+1}$, so that X'' is a projective resolution of M^1.) We may assume by induction that γ' is split, and this allows us to find δ: $H_n(B_1 \times B_2, M_1 \otimes M_2) \to \sum_{l=0}^{n-1} H_{n-l}(B_1, M_1) \otimes H_l(B_2, M_2)$ which splits the restriction of γ. Moreover, $H_0(B_1, M_1) \otimes H_n(B_2, M_2) \to H_n(B_1 \times B_2, M_1 \otimes M_2)$, when followed by δ, is zero. The inductive assumption also allows us to find η': $H_n(B_1 \times B_2, M_1 \otimes M_2) \to H_n(B_1, M_1) \otimes H_0(B_2, M_2)$ which splits the restriction of γ and which gives zero when preceded by $\sum_{l=1}^{n} H_{n-l}(B_1, M_1) \otimes H_l(B_2, M_2) \to H_n(B_1 \times B_2, M_1 \otimes M_2)$. By symmetry, there is η: $H_n(B_1 \times B_2, M_1 \otimes M_2) \to H_0(B_1, M_1) \otimes H_n(B_2, M_2)$ which, together with δ, defines a splitting for γ.

Similarly for cohomology.

Computations for finitely generated abelian groups. Let Z_a be the group of integers modulo a, and suppose $a_m | a_{m-1} | \cdots | a_1$. One shows, for $n \geqq 1$,

$$H_n(Z_{a_1} \times \cdots \times Z_{a_m}, Z) = \sum_{s=1}^{m} \phi(s, n) Z_{a_s}$$

where $\phi(s, n) = \sum_{t=1}^{n} \phi(s-1, t)$ $(s > 1)$, $\phi(1, n) = 1$ if n is odd, 0 otherwise, $lZ_{a_s} = Z_{a_s} \oplus \cdots \oplus Z_{a_s}$ (l times). (The module structure for Z is the trivial one.) The proof is by induction on m, starting from the cyclic group case ($m=1$). Apply (1.3) to $Z_{a_1}\times(Z_{a_2}\times\cdots\times Z_{a_m})$. Similarly, if $Z^l=Z\times\cdots\times Z$ (l times),

$$H_n(Z^l, Z) = \binom{l}{n} Z.$$

Now consider an arbitrary finitely generated abelian group $G=Z_{a_1}\times\cdots\times Z_{a_m}\times Z^l$. Using (1.3), we may combine the two previous results to obtain

$$H_n(G, Z) = \sum_{s=0}^{m} \phi(s, l, n) Z_{a_s}$$

where

$$a_0 = 0, \quad \phi(s,l,n) = \sum_{t=1}^{n} \binom{l}{n-t} \phi(s,t) \ (s, n \geqq 1), \quad \phi(0,l,n) = \binom{l}{n},$$

and $\phi(s, l, 0)=0$ ($s\geqq 1$).

Finally, if G acts trivially on an arbitrary abelian group M, (1.3) yields $H_n(G\times\{e\}, Z\otimes M)=H_n(G, Z)\otimes M\oplus \mathrm{Tor}_1^Z(H_{n-1}(G, Z), M)$, whence

$$H_n(G, M) = \sum_{s=0}^{m} \phi(s, l, n) M_{a_s} \oplus \sum_{s=1}^{m} \phi(s, l, n-1)_{a_s}M,$$

where $M_{a_s}=M/a_sM$, ${}_{a_s}M=\{m\in M \mid a_sm=0\}$. Similarly, using (1.4),

$$H^n(G, M) = \sum_{s=0}^{m} \phi(s, l, n)_{a_s}M \oplus \sum_{s=1}^{m} \phi(s, l, n-1) M_{a_s}.$$

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