OPERATIONS PRESERVING ALL EQUIVALENCE RELATIONS

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In the recent book by R. S. Pierce [**1**, p. 38] the following theorem is given as an exercise. If f is a finitary operation on a set A of cardinal >2 and f has the substitution property relative to every equivalence relation on A, then either f is a constant or f is a projection (that is, there exists an i such that $f(x)=x(i)$ for all x in the domain of f). We will determine all operations f, finitary or not, with this substitution property.

DEFINITION. $|J|$ denotes the cardinal of J. If β is a cardinal, β^+ is the next larger cardinal.

DEFINITION. If R is an equivalence relation on A and $x\in A^\alpha$, $y\in A^\alpha$, then we write xRy whenever $(x(i), y(i))\in R$ for all $i\in\alpha$. An α-ary operation f on A is said to have the substitution property relative to R if $(f(x), f(y))\in R$ whenever xRy.

DEFINITION. If $a\in A$, then R_a denotes the equivalence relation on A whose equivalence classes are $\{a\}$ and $A-\{a\}$.

DEFINITION. If γ is a cardinal, then a filter F of subsets of S is called γ-complete if $\bigcap_{j\in J} C_j\in F$ whenever $|J|<\gamma$ and $C_j\in F$ for all $j\in J$. Every filter is ω-complete.

LEMMA. *Let γ be a cardinal >3 and suppose F is a family of subsets of S. Then F is a γ-complete prime filter if and only if the following condition holds:*

(*) *Whenever $S=\bigcup_{j\in J} B_j$, $|J|<\gamma$, and the B_j are pairwise disjoint, then $B_j\in F$ for exactly one $j\in J$.*

PROOF. If F is a γ-complete prime filter, then the condition is easily seen to hold. Suppose (*) holds. For any $C\in S$, exactly one of C and $S-C$ is in F. Suppose $C\in F$ and $C\subseteq D\subseteq S$. Since $S=C\cup(D-C)\cup(S-D)$, we have $S-D\notin F$. Therefore $D\in F$, and we have shown F is ancestral. Now suppose $C_j\in F$ for all $j\in J$ and J is a well ordered set of cardinal $<\gamma$. If γ is finite we may assume $|J|=2$ and so in all

Presented to the Society, April 21, 1969 under the title *Functions having the substitution property*; received by the editors May 30, 1969 and, in revised form, July 14, 1969.

[1] The preparation of this paper was supported in part by National Science Foundation Grants GP-8827 and GP-9044.

cases, $|J|+1<\gamma$. Let $D_j=S-C_j$ and $D=\bigcup_{j\in J} D_j$. Let $B_j=D_j-\bigcup_{i<j} D_i$. Then $B_j\notin F$ for all $j\in J$, $D=\bigcup_{j\in J} B_j$, and the B_j are pairwise disjoint. Since $S=(S-D)\cup\bigcup_{j\in J} B_j$, it follows from (*) that $S-D\in F$. But $S-D=\bigcap_{j\in J} C_j$, and so the proof is complete.

THEOREM. *Let A be a set of cardinal $\beta>2$, and let f be a nonconstant α-ary operation on A, where α is an ordinal. Then the following are equivalent.*

(i) *f has the substitution property relative to every equivalence relation on A;*

(ii) *f has the substitution property relative to R_a, for all $a\in A$;*

(iii) *there exists a β^+-complete prime filter F of subsets of α such that for all $x\in A^\alpha$, $f(x)=$ the a such that $\{i: x(i)=a\}\in F$.*

REMARK. If α is finite, or if β is infinite and α is less than the first measurable cardinal, the filter F must be principal and so (iii) is equivalent to the assertion that f is a projection.

PROOF. It is obvious that (i) implies (ii). The proof that (ii) implies (iii) is divided into parts. Assume f satisfies the hypothesis of (ii). We may assume that A is identified with β. We first show

(1) For all $x\in A^\alpha$, $f(x)=x(i)$ for some i.

Suppose for some y, $f(y)=a$ and $a\neq y(i)$ for all i. We show f must be constant. Suppose $f(x)=b\neq a$ for some x. There exists $c\in A$ such that $c\neq a$ and $c\neq b$. Let $z\in A^\alpha$ be defined by

$$\begin{aligned} z(i) &= x(i) \qquad \text{if } x(i)\neq a,\\ &= c \qquad \text{otherwise.} \end{aligned}$$

Then zR_bx and therefore $f(z)=b$. But zR_ay, since $z(i)\neq a$ for all i. Hence $(f(z), f(y))=(b, a)\in R_a$, which is a contradiction.

If B is a subset of α, we define the sequence $x_B\in A^\alpha$ by

$$\begin{aligned} x_B(i) &= 0 \qquad \text{for } i\in B,\\ &= 1 \qquad \text{otherwise.} \end{aligned}$$

Let F be the set of all subsets B of α such that $f(x_B)=0$. We show

(2) For any $a\in A$ and any $x\in A^\alpha$, we have $f(x)=a$ if and only if $\{i: x(i)=a\}\in F$.

Let $B=\{i: x(i)=a\}$. There exists $b\in A$ such that $b\neq a$ and $b\neq 0$. Let $y(i)=a$ for $i\in B$, and $y(i)=b$ otherwise. Then xR_ay, and so $f(x)=a$ if and only if $f(y)=a$. Let $z(i)=0$ for $i\in B$, and $z(i)=b$ otherwise. Then yR_bz, and therefore $f(y)\neq b$ if and only if $f(z)\neq b$. By (1), it follows that $f(y)=a$ if and only if $f(z)=0$. But xR_0x_B, and so $f(z)=0$ if and only if $f(x_B)=0$. Hence $f(x)=a$ if and only if $B\in F$.

Now we are ready to prove (iii). By (1) and (2), it is clear that $\emptyset \notin F$. Suppose the family $\langle B_j : j \in J \rangle$ satisfies the hypothesis of (*) in the Lemma, where $S = \alpha$ and $\gamma = \beta^+$. By adding enough copies of the empty set we may assume $J = \beta$. Let $x \in A^\alpha$ be defined by $x(i) = j$ when $i \in B_j$. Then by (2), $B_j \in F$ if and only if $j = f(x)$. Therefore the condition (*) of the Lemma holds and F is a β^+-complete prime filter. By (2) we see that (iii) holds.

To show that (iii) implies (i), let R be any equivalence relation on A. Suppose $x \in A^\alpha$, $y \in A^\alpha$ and xRy. Then since $\{i : x(i) = f(x)\}$ and $\{i : y(i) = f(y)\}$ are in F, there exists $i \in \alpha$ such that $x(i) = f(x)$ and $y(i) = f(y)$. Therefore $(f(x), f(y)) = (x(i), y(i)) \in R$, and the proof is complete.

Reference

1. R. S. Pierce, *Introduction to the theory of absti act algebras*, Holt, Rinehart and Winston, New York, 1968. MR **37** #2655.

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