DIRECT PRODUCT DECOMPOSITION OF COMMUTATIVE SEMISIMPLE RINGS

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ABSTRACT. In this paper it is shown that a commutative semisimple ring is isomorphic to a direct product of fields if and only if it is hyperatomic and orthogonally complete.

In this paper we give a necessary and sufficient condition for a commutative semisimple ring R (i.e., R has no nonzero nilpotent element) to be isomorphic to a direct product of fields. In particular, we show that hyperatomicity and orthogonal completeness is such a necessary and sufficient condition. It is well known that without these conditions R is isomorphic to a subring of a direct product of fields [1, p. 16].

We would like to emphasize that, in what follows, R stands for a commutative ring with no nonzero nilpotent element. Thus, in particular, for every element x of R,

$$x^2 = 0 \quad \textit{if and only if} \quad x = 0. \tag{1}$$

We first prove several lemmas. Lemma 1 below, generalizes the corresponding result for Boolean Rings [2, p. 154].

LEMMA 1. *The ring R is partially ordered by $\leqq$ where for every element x and y of R,*

$$x \leqq y \quad \textit{if and only if} \quad xy = x^2. \tag{2}$$

PROOF. Since $xx = x^2$ it follows from (2) that $x \leqq x$. Thus, $\leqq$ is reflexive.

Moreover, if $x \leqq y$ and $y \leqq x$, then $xy = x^2$ and $yx = y^2$ so that $x^2 - xy - yx + y^2 = (x-y)^2 = 0$. But then $x - y = 0$ by (1). Thus, $x = y$ and therefore $\leqq$ is antisymmetric.

Furthermore, if $x \leqq y$ and $y \leqq z$, then $xy = x^2$ and $yz = y^2$ so that $x^2z = xyz = xy^2 = x^2y = x^3$. Thus, $x^2z^2 = x^3z$ and $x^3z = x^4$ so that $x^2z^2 - x^3z - x^3z + x^4 = (xz - x^2)^2 = 0$. But then from (1) it follows that $xz - x^2 = 0$ or $xz = x^2$. Hence, $x \leqq z$ by (2), and therefore, $\leqq$ is transitive.

In the following the symbol $\leqq$ is used only to represent the partial order defined by (2).

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Let us observe that from (2) it follows immediately that for every element x, y and z of R

$$(3) \qquad x \leqq y \quad \textit{implies} \quad xz \leqq yz$$

and

$$(4) \qquad x^2 = x \quad \textit{implies} \quad xy \leqq y.$$

Motivated by [2, p. 7], we introduce the following definition.

DEFINITION 1. *A nonzero element a of R is called a hyperatom of R if and only if for every element x of R*

$$(5) \qquad x \leqq a \quad \textit{implies} \quad x = 0 \quad \textit{or} \quad x = a$$

and

$$(6) \qquad ax \neq 0 \quad \textit{implies} \quad axs = a$$

for some element s of R.

Next, we prove

LEMMA 2. *Let a be a hyperatom of R. For every element r of R if $ar \neq 0$ then ar is a hyperatom of R.*

PROOF. Let $ar \neq 0$. We show that ar is a hyperatom according to Definition 1.

Since $ar \neq 0$, by (6) we have $ars = a$ for some element s of R. Now, let $x \leqq ar$ then from (3) it follows that $xs \leqq ars$. Hence $xs \leqq a$, and in view of (5) we have

$$xs = 0 \quad \textit{or} \quad xs = a.$$

However, $x \leqq ar$ so that $arx = x^2$ and therefore $rsx^2 = rs(arx) = (rsa)rx = arx = x^2$. Consequently, $(rsx - x)^2 = (rs)^2x^2 - 2rsx^2 + x^2 = 0$. Thus, by (1) we have $rsx = x$ which in view of the above implies that $x = 0$ or $x = ar$. Hence ar satisfies (5).

On the other hand, if $arx \neq 0$ then there exists an element t of R such that $arxt = a$. Thus, $(arx)tr = ar$, so that ar satisfies (6).

In view of the above two cases, we see that ar is a hyperatom of R, as desired.

LEMMA 3. *Let a be a hyperatom of R. Then there exists an element s of R such that as is an idempotent hyperatom of R.*

PROOF. Since $a \neq 0$ it follows from (1) that $a^2 \neq 0$. Thus, by (6), there exists an element s of R such that $a^2s = a$. Clearly, $as \neq 0$ and therefore as is a hyperatom by Lemma 2. But also, $(as)^2 = (a^2s)s = as$. Thus, as is an idempotent hyperatom of R.

DEFINITION 2. *A subset S of R is called orthogonal if and only if $xy=0$ for every two distinct elements x and y of S.*

LEMMA 4. *The set $(e_i)_{i\in I}$ of all idempotent hyperatoms of R is an orthogonal set.*

PROOF. Let e_i and e_j be idempotent hyperatoms of R. From (4) it follows that $e_ie_j \leqq e_i$ and $e_ie_j \leqq e_j$ so that $e_ie_j=e_i=e_j$ or $e_ie_j=0$.

LEMMA 5. *Let $(e_i)_{i\in I}$ be the set of all idempotent hyperatoms of R. Then for every element i of I the ideal*

$$F_i = \{re_i \mid r \in R\} \tag{7}$$

is a subfield of R and

$$F_i \cap F_j = \{0\} \qquad \text{if } i \neq j. \tag{8}$$

PROOF. Since $e_i^2=e_i$ it follows that e_i is an element of F_i and for every element r of R we have $(re_i)e_i=re_i$ so that e_i is the unit of F_i. Further, if $re_i\neq 0$ then since e_i is a hyperatom there exists an element s of R such that $(se_i)(re_i)=sre_i=e_i$. Thus, each nonzero element of F_i has an inverse in F_i so that F_i is a field.

Finally, if $i\neq j$ and $re_i=se_j$ then $re_ie_j=se_je_j=se_j=re_i=0$ since $e_ie_j=0$. Thus, $F_i\cap F_j=\{0\}$.

DEFINITION 3. *The ring R is called hyperatomic if and only if for every nonzero element r of R there exists a hyperatom a of R such that $a\leqq r$.*

In the following sup S, for a subset S of R, is used in reference to the order given by (2).

LEMMA 6. *Let R be hyperatomic and let $(e_i)_{i\in I}$ be the set of all idempotent hyperatoms of R. Then for every nonzero element q of R there exists an idempotent hyperatom e_k such that $qe_k\neq 0$. Moreover, for every element r of R the* $\sup_i re_i$ *exists and*

$$r = \sup_i re_i. \tag{9}$$

PROOF. By Definition 3 if $q\neq 0$ there is a hyperatom a of R such that $a\leqq q$ or $qa=a^2\neq 0$. But then by Lemma 3 there exists an element s of R such that as is an idempotent hyperatom. Clearly, $q(as)\neq 0$. Hence $qe_k\neq 0$ for some idempotent hyperatom $e_k=as$ of R.

To prove the second assertion we note that $r(re_i)=(re_i)^2$ for every element i of I so that r is an upper bound of the set $(re_i)_{i\in I}$. Let u be any upper bound of $(re_i)_{i\in I}$. Then $ure_i=(re_i)^2=rre_i$ for every element i of I. We claim that $r\leqq u$. Assume on the contrary, that $ur-r^2=q\neq 0$. Thus, in view of the above, for some element k of I it is the case that $ure_k - r^2 e_k = qe_k \neq 0$. But this contradicts that $ure_i = rre_i$ for every element i of I. Thus, indeed $r = \sup_i re_i$. Hence, the lemma is proved.

If $(e_i)_{i \in I}$ is the set of all idempotent hyperatoms of R, then by (8) we may consider the direct product $\prod_i F_i$ of the fields F_i given by (7). We then have the following

LEMMA 7. *Let R be hyperatomic and let $(e_i)_{i \in I}$ be the set of all idempotent hyperatoms of R. Then*

$$\alpha(r) = (re_i)_{i \in I} \tag{10}$$

is an isomorphism from R onto a subring of the direct product $\prod_i F_i$ of fields F_i.

PROOF. Clearly, α is a homomorphism so it suffices to show that α is one-to-one. But if $r \neq q$ it follows from (9) that $\sup_i re_i \neq \sup_i qe_i$ so that $(re_i)_{i \in I} \neq (qe_i)_{i \in I}$ or $\alpha(r) \neq \alpha(q)$. Hence, α is an isomorphism.

As mentioned earlier the existence of an isomorphism from R to a subring of a direct product of fields is well known and requires no additional conditions on R. However, the special isomorphism α given in Lemma 7 is essential for the proof of the theorem below which states that hyperatomicity and orthogonal completeness of R is a necessary and sufficient condition for R to be isomorphic to a direct product of fields.

One additional lemma is also needed, but first we observe that if $\sup_i r_i$ exists for a subset $(r_i)_{i \in I}$ of R then $r_i \leqq \sup_i r_i$, so that by (2) we have

$$r_i \sup_i r_i = r_i^2. \tag{11}$$

LEMMA 8. *Let $(r_i)_{i \in I}$ be a subset of R such that* $\sup_i r_i$ *exists. Then for every element b of R the* $\sup_i br_i$ *exists and*

$$b \sup_i r_i = \sup_i br_i. \tag{12}$$

PROOF. From (11) it follows that $(br_i)(b \sup_i r_i) = (br_i)^2$ so that $br_i \leqq b \sup_i r_i$ for every element i of I. Thus, $b \sup_i r_i$ is an upper bound of the set $(br_i)_{i \in I}$.

Let u be any upper bound of $(br_i)_{i \in I}$. Then $ubr_i = (br_i)^2$ for every element i of I. Thus, $ubr_i - b^2 r_i^2 + r_i^2 = r_i^2$. But then from (11) we see that for every element i of I

$$r_i\left(ub - b^2 \sup_i r_i + \sup_i r_i\right) = r^2$$

so that

$$r_i \leqq ub - b^2 \sup_i r_i + \sup_i r_i$$

and therefore

$$\sup_i r_i \leqq ub - b^2 \sup_i r_i + \sup_i r_i.$$

Consequently, it follows from (2) that

$$\left(\sup_i r_i\right)\left(ub - b^2 \sup_i r_i + \sup_i r_i\right) = \left(\sup_i r_i\right)^2$$

which implies that

$$ub \sup_i r_i = \left(b \sup_i r_i\right)^2$$

and therefore,

$$b \sup_i r_i \leqq u.$$

Hence, $\sup_i br_i$ exists and is equal to $b \sup_i r_i$.

DEFINITION 4. *The ring R is called orthogonally complete if and only if* sup *S of every orthogonal subset S of R exists.*

Finally, we have

THEOREM. *The ring R is isomorphic to a direct product of fields if and only if R is hyperatomic and orthogonally complete.*

PROOF. Let β be an isomorphism from R onto a direct product $\prod_{i\in I} K_i$ of fields K_i. Let r be a nonzero element and let $\beta(r) = (r_i)_{i\in I}$. There exists an element j of I such that $r_j \neq 0$. Let u_j be the unit of K_j. The element a of R given by $a = r\beta^{-1}((k_i)_{i\in I})$ with $k_j = u_j$ and $k_i = 0$ for $i \neq j$ is obviously a hyperatom of R with $a \leqq r$. Hence, R is hyperatomic.

Next, let S be an orthogonal subset of R and let $\beta[S] = ((k_i(s))_{i\in I})_{s\in S}$. Since S is orthogonal it follows that $\beta^{-1}((k_i)_{i\in I}) = \sup S$ where $k_i = k_i(s)$ if $k_i(s) \neq 0$ for some element s of S and $k_i = 0$ otherwise. Thus, R is orthogonally complete.

Conversely, if R is hyperatomic and orthogonally complete we show that R is isomorphic to the direct product $\prod_{i\in I} F_i$ of fields F_i given by (7). In view of Lemma 7 it will suffice to show that the mapping α of (10) is an onto mapping. Let $(r_ie_i)_{i\in I}$ be any element of $\prod_i F_i$. By Lemma 4 the set $(r_ie_i)_{i\in I}$ is an orthogonal subset of R. Let $h = \sup_i r_ie_i$. But then from (12) and Lemma 4 it follows that for every element j of I

$$he_j = e_j \sup_i r_i e_i = \sup_i r_i e_i e_j = r_j e_j.$$

Hence, by (10) we have $\alpha(h) = (he_i)_{i \in I} = (r_i e_i)_{i \in I}$ so that α is an onto mapping. Thus, the theorem is proved.

References

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